

SUPPLEMENTARY TOPICS FOR MATH 106

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SECTION 1.9 – INVERSE FUNCTIONS AND LOGARITHMS

1 – Inverse Functions.

We know that for every member of the domain of a function there is exactly one member of the range. For some functions, the reverse is true. That is, every member of the range corresponds to exactly one member of the domain. When a function has this property, we define the inverse of that function to take each member of the range "back to where it came from".

For example, the cube function has this one-to-one property and the cube root function is its inverse. The square function does not have the property since 2 and -2 have the same square. However, if we restrict this function to nonnegative numbers, the inverse is the square root function. Notice that the cube of 2 is 8 and the cube root of 8 is 2. Likewise, taking cube of the cube root of 4 gives back 4. (You know this without even knowing what is the cube root of 4.) Indeed, every function with an inverse has this property. Applying the function and its inverse in either order gives back what you started with. (Who is my wife's husband?)

2 – Logarithms.

In section 1.7, we examined the exponential function for a fixed base. For a base a not equal to 1, this function has an inverse, called the logarithm to the base a (denoted \log_a). For example, $\log_2 8 = 3$. We don't know exactly $\log_2 12$, but we know that it is between 3 and 4. Notice that logarithms are only defined for positive numbers. For $a > 1$, a^x grows very fast for positive x . The reverse is true for $\log_a x$, it grows slowly for $x > 1$.

Logarithms inherit some important properties from those of exponents that we saw in section 1.5.

- a) $\log_a 1 = 0$;
- b) $\log_a a = 1$;
- c) $\log_a(uv) = \log_a u + \log_a v$;
- d) $\log_a(u^p) = p \log_a u$.

The most common base for logarithms is 10. That is the one given by the log key on your calculator. The next most common is e . That is the one given by the ln key. Unless we say otherwise, we will assume that we are using base 10.

3 – Applications.

1. Many measures that you encounter use logarithms. The Richter magnitude for earthquakes is determined by the log of the intensity of the quake. Hence the intensity is proportional to 10 raised to the magnitude. Each increase in the magnitude by 1 corresponds to an increase in the intensity by a factor of 10. The magnitude of the smallest detectable quake is 0. So the magnitude of a quake a million times as intense is 6. So Richter magnitudes are confined to a much smaller range of values than the intensities. But be aware that a small change in magnitude represents a relatively large change in intensity. For example,

$$10^{4.9} = (10^{-2})(10^{4.7}) = 1.58(10^{4.7}).$$

So a 4.9 quake is 58% larger than a 4.7.

2. pH is a measure of acidity. It is $-\log H$ where H is the hydrogen ion concentration. This concentration ranges from about 1 (acidic) to 10^{-14} (alkaline), so pH ranges from 0 to 14. $pH = 7$ is considered neutral.

If $pH = 6.7$, then concentration is

$$10^{-6.7} = (10^{-3})(10^{-7}) = 2(10^{-7}),$$

so has twice the ion concentration as neutral.

Decibels and stellar magnitudes also are defined in terms of logarithms.

Before calculators were widely available, slide rules were used to multiply two numbers by adding their logs. Now they can be used with calculators to solve equations which could not be solved with calculators alone. These methods will be particularly useful in chapter 6.

Examples

Solve each of the following for x .

1. $x = 3(1.5^{2.3})$

Solution: We don't need logarithms for this one. Use the calculator to find $1.5^{2.3} = 2.541$ and then $x = 3(2.541) = 7.623$.

2. $2^x = 12$

Solution: We are looking for the $\log_2 12$. The calculator doesn't give this directly. However, taking the log (to base 10) of both sides gives

$$x \log 2 = \log 12.$$

So

$$x = \frac{\log 12}{\log 2} = 3.585.$$

Check that $2^{3.585} = 12$.

$$3. 4.2 = 6.1(3.7^x)$$

Solution: First divide both sides by 6.1 and take the log of both sides:

$$\log\left(\frac{4.2}{6.1}\right) = \log(3.7^x) = x \log(3.7).$$

But $\log(.6885) = -.162$, and $\log(3.7) = .568$. Thus

$$-.162 = x(.568);$$

so

$$x = -\frac{.162}{.568} = -.285.$$

As always, check your answer by plugging it in to the equation.

$$4. 7.3 = 2.3(x^{1.8})$$

Solution: Divide both sides by 2.3 and take the log of both sides:

$$1.8 \log x = \log\left(\frac{7.3}{2.3}\right) = .5016.$$

Thus

$$\log x = \frac{.5016}{1.8} = .279;$$

so $x = 10^{.279} = 1.9$. Alternatively,

$$x = \left(\frac{7.3}{2.3}\right)^{1/1.8} = 1.9.$$

Exercises

1. Evaluate the following without the aid of a calculator.

a) $10^{\log 3}$

b) $\log(10^{1.7})$

c) $\log(.001)$

d) $\log[.01(10^{-5})]$

2. Expand or simplify the following.

a) $\log(100x^3)$

b) $\log(x^2y^3)$

c) $100^3 \log(2x)$

d) $.1^x \log 4$

3. Solve each for x . Check each of your answers.

a) $x = 7.2(.75^{6.4})$

b) $6.4 = 3.8(x^{2.3})$

c) $5.7 = 2.7(x^{.8})$

d) $1.8 = 3.8(2.6^x)$

e) $3.6 = 1.3(.6^x)$

4. An earthquake with a Richter magnitude of 3.9 is how many times as intense as one with a 3.6 magnitude?

5. How do the ion concentrations of solutions compare if their pH 's are 7.2 and 7.6?