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Convergence of periodically forced rank-type equations

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Consider a difference equation which takes the k-th largest output of m functions of the previous m terms of the sequence. If the functions are also allowed to change periodically as the difference equation evolves, this is analogous to a differential equation with periodic forcing. A large class of such non-autonomous difference equations are shown to converge to a periodic limit, which is independent of the initial condition. The period of the limit does not depend on how far back each term is allowed to look back in the sequence, and is in fact equal to the period of the forcing.

Keywords: max-type equations; rank-type equations; periodic solutions; non-autonomous equations

1. Introduction

Recently, there has been substantial interest in max-type difference equations [1–9, 14–18]. For the equation

$$x_n = \max_{1\leq i\leq M}\{f_i(x_{n-i})\}, \tag{1}$$

with initial sequence $(x_1, \ldots, x_M)$, it has been shown [11] that if the f_i are contractive, then all solutions converge to a fixed point, i.e.

$$\lim_{n\to\infty} x_n = r_*.$$

Moreover, the fixed point r_* can be identified as the maximum of the individual fixed points

$$r_* = \max_{1\leq i\leq M}\{r_i\},$$

where r_i is the unique fixed point $r_i = f_i(r_i)$.

In view of this result, it is reasonable to ask about convergence in the periodically forced case. Assume that $f_1, \ldots, f_M$ vary periodically with the discrete time variable n. We will investigate the limiting behaviour of the non-autonomous difference equation

$$x_n = \max_{1\leq i\leq M}\{f_i(x_{n-i}, n)\}, \tag{2}$$

where $f_i : \mathbb{R}\times\mathbb{N}\to\mathbb{R}$ are contractions with respect to the first variable and P-periodic with respect to the second. That is, we assume there exists $\alpha < 1$ such that

$$|f_i(x,n) - f_i(y,n)| \leq \alpha|x-y|, \tag{3}$$

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for all *i, n* and $f_i(x, n+P) = f_i(x, n)$ for all *n*. We think of *P* as the *forcing period* and *M* as the *memory length*. We will show below that solutions of the contractive, *P*-periodic difference equation (2) converge to a unique periodic orbit with period *P*, for all initial sequences.

In [10], rank-type equations were proposed as a generalization of max-type equations. Consider the difference equation

$$x_n = \underset{1\le i\le M}{k\text{-rank}}\{f_i(x_{n-i})\}, \tag{4}$$

where *k*-rank denotes the *k*th largest value. It was shown in [10] that with the same contractiveness hypotheses, solutions converge to $r_* = k\text{-rank}\{r_i\}$, the *k*th largest of the individual fixed points. In this paper, we further generalize this result to the periodically forced case. Our main result is the following:

THEOREM 1.1. *Let $f_1, \ldots, f_M : \mathbb{R}\times\mathbb{N}\to\mathbb{R}$ be contractions with respect to the first variable and P-periodic with respect to the second. Let $1 \le k \le M$. Then, for any initial sequence, the solution of the difference equation*

$$x_n = \underset{1\le i\le M}{k\text{-rank}}\{f_i(x_{n-i}, n)\}, \tag{5}$$

is asymptotically periodic with period P.

Unlike the autonomous case, in general, we do not know how to find a formula for the periodic solution in terms of the individual dynamics of the f_i. In the fourth section of this article, we relate some partial progress in this direction.

The conclusion of Theorem 1.1 fails if $\alpha \ge 1$ in (3), even if $k = P = 1$, the autonomous max-type case [7]. We mention two well-known examples where the condition on α does not hold.

Example 1.2. Consider the difference equation

$$x_n = \max\{-x_{n-1}, -x_{n-2}\}.$$

Although this is an autonomous max-type equation ($k = P = 1$), it is easy to check that all solutions have period $3 \ne P$. In this example, $\alpha = 1$, and Theorem 1.1 does not apply.

Example 1.3. The first-order difference equation

$$x_n = \max\{1 - 2x_{n-1}, 2x_{n-1} - 1\}$$

has bounded, non-periodic (chaotic) solutions for almost every initial condition x_0 between 0 and 1. In this example, $\alpha = 2$. As a dynamical system, this example is the upside-down tent map.

Example 1.4. For a straightforward application of Theorem 1.1, fix positive integers *P* and $k \le M$, and denote by *A* and *B* two $P \times M$ matrices of real numbers, where the entries $|A_{ij}| < 1$. Define the difference equation

$$x_n = k\text{-rank}\{A_{\bar{n}1}x_{n-1} + B_{\bar{n}1}, A_{\bar{n}2}x_{n-2} + B_{\bar{n}2}, \ldots, A_{\bar{n}M}x_{n-M} + B_{\bar{n}M}\}, \tag{6}$$

where $\bar{n} \equiv 1 + (n - 1 \bmod P)$. Thus, the coefficients cycle through the rows of the matrices A and B, forcing the equation with period P. Theorem 1.1 implies that all solutions converge asymptotically to a period P solution.

Example 1.5. Let A be a $P \times M$ matrix of positive real numbers and consider the recurrence

$$x_n = \max\left\{A_{\bar{n}1}x_{n-1}^{\alpha_1}, \ldots, A_{\bar{n}M}x_{n-M}^{\alpha_m}\right\}, \tag{7}$$

where $-1 < \alpha_i < 1$ and $\bar{n} = 1 + (n - 1 \bmod P)$. It follows from Theorem 1.1 (applied to $y_n = \ln x_n$) that the recurrence converges to a unique period P orbit. Note that this result is independent of the memory length M. The periodicity of the limit only depends on the periodicity of the forcing.

In sufficiently simple cases, we can say more about the convergent solution. In Section 4, we further pursue the special case $M = P = 2$ of (7). A closed form for the globally attracting solution of (7) is given by

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \max\left\{A_{12}A_{21}^{(\alpha_1/1-\alpha_2)}, A_{11}^{(\alpha_1/1-\alpha_1^2)}A_{12}^{(1/1-\alpha_1^2)}, A_{22}^{(1/1-\alpha_2)}\right\}\\
\lim_{n\to\infty} x_{2n+1} &= \max\left\{A_{11}A_{22}^{(\alpha_1/1-\alpha_2)}, A_{12}^{(\alpha_1/1-\alpha_1^2)}A_{11}^{(1/1-\alpha_1^2)}, A_{21}^{(1/1-\alpha_2)}\right\}.
\end{aligned}$$

(See Example 4.5 for details.) The complexity of this solution contrasts with the simplicity of the case $P = 1$, which is simply $x_n \longrightarrow \max\left\{A_i^{(1/1-\alpha_i)}\right\}$ (see [11–13]).

Example 1.6. Let A_1, A_2, A_3 be real numbers less than 0.15, B_1, B_2, B_3 be arbitrary real numbers and P be a positive integer. Then, it follows from Theorem 1.1 that every solution of

$$x_n = \mathrm{median}\left\{\mathrm{e}^{A_1\sin(B_1+2\pi n/P)-x_{n-1}^2}, \mathrm{e}^{A_2\sin(B_2+2\pi n/P)-x_{n-2}^2}, \mathrm{e}^{A_3\sin(B_3+2\pi n/P)-x_{n-3}^2}\right\},$$

is asymptotically periodic with period P. Note that the median is synonymous with 2-rank. It is easily checked that the condition $A_i < 0.15 < (1/2) - (\ln 2/2)$ implies that the functions $f_i(x) = \exp(A_i \sin(B_i + 2\pi n/P) - x^2)$ are contractive, so the convergence is implied by Theorem 1.1.

Our main convergence result Theorem 1.1 will be proved as a special case of a more general result, Corollary 3.5, which applies to a class of difference equations called sup-contractive. The next example is covered under the sup-contractive hypothesis, which is more general than a fixed k-rank.

Example 1.7. Let $f_1, \ldots, f_M : \mathbb{R} \longrightarrow \mathbb{R}$ be contractions. Let $P \leq M$ be a positive integer and denote $\bar{n} = 1 + (n \bmod P)$. We will show in Example 3.7 that every solution of

$$x_n = \frac{1}{2}\left(\max_{1\leq i\leq M}\{f_i(x_{n-i})\} - \bar{n}\text{-}\mathrm{rank}_{1\leq i\leq M}\{f_i(x_{n-i})\}\right),$$

is asymptotically periodic with period P.

In Section 2 below, we develop some facts about functions that are contractive in the sup-norm. An important result is Lemma 2.4, which will imply that (5) is sup-contractive. In Section 3, we show that the general class of sup-contractive recurrence equations converges as desired. Finally, in Section 4, we return to the max-type equation (2), and find the value of the asymptotically convergent periodic orbit for some specific values of M and P.

2. Sup-contractive functions

The convergence proofs in the next section apply to a wide class of functions $G : \mathbb{R}^a \to \mathbb{R}^b$, which includes compositions of contractive functions as in (3) with the max and k-rank functions. We will be interested in the operator norm of functions G, where the norm used is the sup-norm. Namely, assume there exists $L \geq 0$ such that

$$\|G(x) - G(y)\|_\infty \leq L\|x - y\|_\infty,$$

for all $x, y \in \mathbb{R}^a$. We can think of L as the Lipschitz constant of G in the sup-norm, and will refer to it as the *sup-Lipschitz constant* in the following sections. When $L = 1$, we will call G *sup-non-expansive* and when $L < 1$ we will call G *sup-contractive*. Note that a contraction on $\mathbb{R}$ is sup-contractive since the infinity norm on $\mathbb{R}$ is simply the absolute value.

Example 2.1. Let $G : \mathbb{R}^M \to \mathbb{R}$ be the function $G(x) = c\|x\|_p$, where $c > 0$. Since

$$|c\|x\|_p - c\|y\|_p| = c|\|x\|_p - \|y\|_p| \leq c\|x - y\|_p \leq cM^{1/p}\|x - y\|_\infty,$$

G is sup-non-expansive when $c = M^{-1/p}$ and sup-contractive when $c < M^{-1/p}$.

Example 2.2. For a fixed $z \in \mathbb{R}^M$, define $G : \mathbb{R}^M \to \mathbb{R}$ by $G(x) = z^Tx$. Since

$$|z^Tx - z^Ty| = | z^T(x - y)| = \sum_{i=1}^{M} |z_i||x_i - y_i| \leq \|x - y\|_\infty \sum_{i=1}^{M} |z_i| = \| x - y\|_\infty \|z\|_1,$$

G is sup-non-expansive when $\|z\|_1 = 1$, and sup-contractive when $\|z\|_1 < 1$.

Example 2.3. For $1 \leq k \leq M$, let $R_k : \mathbb{R}^M \to \mathbb{R}$ be the function that returns the k-th largest entry of the M-dimensional input vector. Customarily, R_k is called the 'k-rank' function. It includes the max ($k = 1$) and min ($k = M$) as special cases. Not surprisingly, the max function is sup-non-expansive. Somewhat more surprising is that this property holds for all k, as shown in the next lemma.

LEMMA 2.4. *The k-rank function R_k is sup-non-expansive.*

Proof. Let $x, y \in \mathbb{R}^M$, and assume, without loss of generality, that $R_k(y) \leq R_k(x)$. Set $y_{r(i)}$ be the i-th largest component of y and let $x_{s(i)}$ be the i-th largest component of x. Then,

$$y_{r(1)} \leq y_{r(2)} \leq \cdots \leq y_{r(k)} = R_k(y) \leq R_k(x) = x_{s(k)} \leq x_{s(k+1)} \leq \cdots \leq x_{s(M)}.$$

Now, examine the list of natural numbers $I = (r(1), \ldots, r(k), s(k), \ldots, s(M))$. We note that I has length $M + 1$ but each entry is chosen from $\{1, \ldots, M\}$. Thus, by the pigeonhole principle, at least two of the listed numbers must be the same. Note that all the $r(1), \ldots, r(k)$ are distinct and all the $s(k), \ldots, s(M)$ are distinct, thus there must be some

$1 \leq i \leq k$ and some $k \leq j \leq M$ such that $r(i) = s(j) = t$. Therefore,

$$y_t = y_{r(i)} \leq R_k(y) \leq R_k(x) \leq x_{s(j)} = x_t,$$

which implies that

$$|R_k(x) - R_k(y)| = R_k(x) - R_k(y) \leq x_t - y_t = |x_t - y_t| \leq \|x - y\|_\infty.$$

Since the absolute value is the infinity norm on $\mathbb{R}$, R_k is sup-non-expansive. □

The sup-Lipschitz constants are multiplied under composition, based on the next lemma.

LEMMA 2.5. *Assume $f_1, \ldots, f_k : \mathbb{R}^M \to \mathbb{R}$ have sup-Lipschitz constants less than L_1 and $f : \mathbb{R}^k \to \mathbb{R}$ has sup-Lipschitz constant L_2, then $g(x) = f(f_1(x), f_2(x), \ldots, f_k(x))$ has sup-Lipschitz constant no larger than $L_1 L_2$.*

Proof. Let $x, y \in \mathbb{R}^M$ then

$$\begin{aligned} |g(x) - g(y)| &= |f(f_1(x), \ldots, f_k(x)) - f(f_1(y), \ldots, f_k(y))| \\ &\leq L_2 \max_{1 \leq i \leq k} |f_i(x) - f_i(y)| \leq L_1 L_2 \|x - y\|_\infty. \end{aligned}$$

□

Note that if f is sup-contractive, and all f_i are sup-non-expansive then g is sup-contractive. Similarly, if f is sup-non-expansive, and all f_i are sup-contractive then g is sup-contractive. Finally, the next lemma shows that if we bundle together functions into a vector, the sup-Lipschitz constant cannot grow.

LEMMA 2.6. *Assume $f_1, \ldots, f_k : \mathbb{R}^M \to \mathbb{R}$ have sup-Lipschitz constants at most L. Then, $f : \mathbb{R}^M \to \mathbb{R}^k$ defined by*

$$f(x) = (f_1(x), f_2(x), \ldots, f_k(x)),$$

has sup-Lipschitz constant at most L.

Proof. Let $x, y \in \mathbb{R}^M$, then

$$\|f(x) - f(y)\|_\infty = \max_{1 \leq i \leq k} \{|f_i(x) - f_i(y)|\} \leq L\|x - y\|_\infty.$$

□

Note that if $k = M$ and $L < 1$, then in fact f is a contraction on $\mathbb{R}^M$, so bundling functions is a way to build contractions. Finally, note that these statements could be generalized to allow f_i to be vector valued but this is not necessary for what follows.

3. General case

We now return to our original recurrence equation

$$x_n = \underset{1 \leq i \leq M}{k\text{-rank}} \{f_i(x_{n-i}, n)\},$$

under the assumption that each $f_i : \mathbb{R} \to \mathbb{R}$ is a contraction. Since k-rank is sup-non-expansive by Lemma 2.4, and each f_i is sup-contractive, the composition of the two is sup-contractive by Lemma 2.5. This is a special case of the equation

$$x_n = G_n(x_{n-1}, \dots, x_{n-M}),$$

where $G_1, \dots, G_P$ are any sup-contractive functions and $G_{n+P} = G_n$ for all n. Setting $s = PM$, we will first show that for all initial conditions, the limit is periodic of period s by a contraction mapping argument. We will then use the P-periodicity of G_n to show that the limit is actually periodic of period P.

With slight abuse of notation, we can consider G_n as a function of all s variables, although it depends only on the first M

$$x_n = G_n(x_{n-1}, \dots, x_{n-s}) = G_n(x_{n-1}, \dots, x_{n-M}), \tag{8}$$

where $G_{n+s} = G_{n+MP} = G_n$ for all n. We define

$$\bar{x}_n = (x_{ns+1}, \dots, x_{ns+s}),$$

and let $\bar{x}_0$ be the initial condition. We want to show that we can write

$$\bar{x}_{n+1} = \bar{F}(\bar{x}_n),$$

where $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$ is a contraction in the infinity norm on $\mathbb{R}^s$. Let $\bar{y} = (y_1, \dots, y_s)$ and define the following functions for $k = 1, \dots, s$.

$$\begin{aligned} F_1(\bar{y}) &= G_1(y_s, y_{s-1}, \dots, y_1) \\ F_k(\bar{y}) &= G_k(F_{k-1}(\bar{y}), \dots, F_1(\bar{y}), y_s, y_{s-1}, \dots, y_k). \end{aligned}$$

Note that this is an inductive definition since F_k depends on $F_1, \dots, F_{k-1}$. Finally, we define

$$\bar{F}(\bar{y}) = (F_1(\bar{y}), \dots, F_s(\bar{y})). \tag{9}$$

The recursive nature of the definition of $\bar{F}$ requires the following lemma to show that $\bar{F}$ encapsulates the evolution of the sequence $\{x_n\}$.

LEMMA 3.1. *Let $G_1, \dots, G_s : \mathbb{R}^s \to \mathbb{R}$ and let $\{x_i\}$ be defined by* (8). *Define $\bar{F}$ by* (9). *Then, for all n, we have $\bar{x}_{n+1} = \bar{F}(\bar{x}_n)$.*

Proof. This is equivalent to showing that $x_{ns+s+k} = F_k(\bar{x}_n)$ for $k = 1, \dots, s$ which is equivalent to

$$F_k(\bar{x}_n) = G_k(x_{ns+s+k-1}, \dots, x_{ns+k}).$$

Note that when $k = 1$, we have

$$F_1(\bar{x}_n) = G_1(x_{ns+s}, \dots, x_{ns+1}) = x_{ns+s+1},$$

by definition of F_1 and G_1. Then, for $2 \le k \le s$, we can proceed by induction. Assuming $x_{ns+s+\bar{k}} = F_{\bar{k}}(\bar{x}_n)$ for all $\bar{k} < k$, we have

$$\begin{aligned} F_k(\bar{x}_n) &= G_k(F_{k-1}(\bar{x}_n), \dots, F_1(\bar{x}_n), x_{ns+s}, \dots, x_{ns+k}) \\ &= G_k(x_{ns+s+k-1}, \dots, x_{ns+s+1}, x_{ns+s}, \dots, x_{ns+k}) = x_{ns+s+k}, \end{aligned}$$

which completes the proof. □

Now that we have defined $\bar{F}$, we need to show that it is a contraction.

THEOREM 3.2. *Let $G_1, \ldots, G_s : \mathbb{R}^s \to \mathbb{R}$ be sup-contractive and define $\bar{F}$ as in* (9). *Then, $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$ is a contraction with respect to the infinity norm.*

Proof. By Lemma 2.6, it suffices to show that each F_i is sup-contractive. Note that the projection function $\pi_i(\bar{y}) = y_i$ is sup-non-expansive since

$$|\pi_i(\bar{x}) - \pi_i(\bar{y})| = |x_i - y_i| \leq \|\bar{x} - \bar{y}\|_\infty.$$

Thus, $F_1 = G_1(\pi_s(\bar{y}), \ldots, \pi_1(\bar{y}))$ is sup-contractive by Lemma 2.5, because G_1 is sup-contractive and π_i is sup-non-expansive. For $1 < i \leq s$, we proceed by induction. Assume F_j is sup-contractive for all $j < i$, then

$$F_i(\bar{y}) = G_i(F_{i-1}(\bar{y}), \ldots, F_1(\bar{y}), \pi_s(\bar{y}), \pi_{s-1}(\bar{y}), \ldots, \pi_i(\bar{y})).$$

So, F_i is a composition of the sup-contractive function G_i with sup-contractive functions $F_{i-1}, \ldots, F_1$ and sup-non-expansive projection functions $\pi_s, \ldots, \pi_i$. Thus by Lemma 2.5, F_i is sup-contractive, which completes the induction and thus shows that $\bar{F}$ is a contraction. □

COROLLARY 3.3. *Let $G_1, \ldots, G_P : \mathbb{R}^M \to \mathbb{R}$ be sup-contractive and let $G_{n+P} = G_P$ for all n. Given an initial condition $(x_1, \ldots, x_M)$, let*

$$x_n = G_n(x_{n-1}, \ldots, x_{n-M}),$$

then x_n converges to a unique PM-periodic orbit independent of initial conditions.

Proof. By Theorem 3.2, we can construct a contraction mapping $\bar{F} : \mathbb{R}^s \to \mathbb{R}^s$ such that $\bar{x}_{n+1} = \bar{F}(\bar{x}_n)$. Thus, by the contraction mapping theorem, $\bar{F}$ has a unique fixed point $x^* \in \mathbb{R}^s$ and for any $\bar{x}_0 \in \mathbb{R}^s$ we have $\lim_{n\to\infty}\bar{x}_n = x^*$. □

Note that s may not be the prime period (smallest possible period). So, we now show that in fact x_n must be asymptotically periodic of period P.

THEOREM 3.4. *Let $G_1, \ldots, G_P : \mathbb{R}^M \to \mathbb{R}$ be sup-contractive and define $\bar{F}$ as in* (9). *Let x^* be the unique fixed point of $\bar{F}$. Then, x^* is periodic of period P.*

Proof. Define a shift operator by $S(\bar{x}_n) = (x_{ns+1+P}, \ldots, x_{ns+s+P})$. We will use the fact that G_i is actually periodic of period P to show that the function S commutes with $\bar{F}$. Note that

$$\begin{aligned} S(\bar{F}(\bar{x}_n)) &= S(\bar{x}_{n+1}) = (x_{ns+s+1+P}, \ldots, x_{ns+s+s+P}) \\ \bar{F}(S(\bar{x}_n)) &= \bar{F}(x_{ns+1+P}, \ldots, x_{ns+s+P}). \end{aligned}$$

First, we examine the first component of $\bar{F}(S(\bar{x}_n))$

$$\begin{aligned} (\bar{F}(S(\bar{x}_n)))_1 &= G_1(x_{ns+s+P}, \ldots, x_{ns+1+P}) \\ &= G_{1+P}(x_{ns+s+P}, \ldots, x_{ns+1+P}) = x_{ns+s+1+P}, \end{aligned}$$

where the second equality comes from the fact that G_i is periodic of period P. This shows that the first components of $S(\bar{F}(\bar{x}_n))$ and $\bar{F}(S(\bar{x}_n))$ are the same. We proceed inductively to show that all the components are the same. Assume that $(S(\bar{F}(\bar{x}_n)))_j = (\bar{F}(S(\bar{x}_n)))_j$ for all

$j < i$. Then,

$$\begin{aligned}(\bar{F}(S(\bar{x}_n)))_i &= G_i(F_{i-1}(S(\bar{x}_n)), \ldots, F_1(S(\bar{x}_n)), x_{ns+s+P}, \ldots, x_{ns+s+P+i-s}) \\ &= G_{i+P}(x_{ns+s+i-1+P}, \ldots, x_{ns+s+1+P}, x_{ns+s+P}, \ldots, x_{ns+s+P+i-s}) \\ &= x_{ns+s+i+P},\end{aligned}$$

where we have again used the P-periodicity if G_i to conclude $G_i = G_{i+P}$. This shows that

$$S(\bar{F}(\bar{x}_n)) = \bar{F}(S(\bar{x}_n)).$$

So, inductively, we have $S(\bar{x}_n) = \bar{F}^n(S(\bar{x}_0))$. Let x^* be the unique fixed point of $\bar{F}$ and define two sequences, the first with $x_0 = x^*$ and the second with $y_0 = S(x^*)$. Note that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \bar{F}^n(\bar{y}_0) = x^*,$$

since all initial conditions converge to x^*, and at the same time

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \bar{F}^n(\bar{y}_0) = \lim_{n\to\infty} \bar{F}^n(S(x_0)) = \lim_{n\to\infty} \bar{S}(\bar{F}_n(x_0)) = \lim_{n\to\infty} S(x_0) = S(x^*).$$

So, we conclude that $S(x^*) = x^*$ and thus x^* is periodic of period P. □

COROLLARY 3.5. *Let* $G_1, \ldots, G_P : \mathbb{R}^M \to \mathbb{R}$ *be sup-contractive and let* $G_{n+P} = G_P$ *for all* n. *Given an initial condition* $(x_1, \ldots, x_M)$, *let*

$$x_n = G_n(x_{n-1}, \ldots, x_{n-M}),$$

then x_n *converges to a unique P-periodic orbit independent of initial conditions.*

Proof. By Theorem 3.4, there exists a unique $x^* \in \mathbb{R}^s$ which is P-periodic such that $\lim_{n\to\infty} \bar{x}_n = x^*$. Thus, x_n is asymptotically periodic of period P. □

We conclude that x_n approaches a unique periodic orbit, for any initial condition, whose period is equal to the forcing period P. The periodicity of the rank-type equation (5) is now an easy corollary.

COROLLARY 3.6. *For* $i = 1, \ldots, M$, *let* $f_i(x, n) : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ *be contractive in* x *and P-periodic in* n. *Given an initial condition* $(x_1, \ldots, x_M)$ *and* $k \in \{1, \ldots, M\}$, *let*

$$x_n = k\text{-rank}\{f_i(x_{n-i}, n)\},$$

then x_n *converges to a unique P-periodic orbit independent of initial conditions.*

Proof. Recall that by Lemma 2.4, the k-rank function is sup-non-expansive, and each $f_i(x, n)$ is sup-contractive in x so by Lemma 2.5, the composition is sup-contractive. By Corollary 3.5, x_n is asymptotically periodic of period P. □

We can now return to the equation from Example 1.7.

Example 3.7. Let $f_1, \dots, f_M : \mathbb{R} \to \mathbb{R}$ be contractions. Let $P \leq M$ be a positive integer and set $\bar{n} = 1 + (n \bmod P)$. Let

$$x_n = G_n(x_{n-1}, \dots, x_{n-M}) = \frac{1}{2}\left(\max\{f_i(x_{n-i})\} - \bar{n}\text{-rank}\{f_i(x_{n-i})\}\right)$$

Recall that by Lemma 2.4, the rank functions are all sup-non-expansive. Thus, $\bar{n}$-rank$\{f_i(x_{n-i})\}$ is a composition of a sup-non-expansive function with sup-contractive functions f_i, and thus the composition is sup-contractive by Lemma 2.5. Furthermore, setting $\bar{z} = (1/2, -1/2)$, we see that $\|\bar{z}\|_1 = 1$ so $f(x) = z^T x$ is sup-non-expansive. Therefore,

$$\begin{aligned} G_n(y_1, \dots, y_M) &= z^T\left(\max\{f_i(y_i)\}, \bar{n}\text{-rank}\{f_i(y_i)\}\right) \\ &= \frac{1}{2}\left(\max\{f_i(y_i)\} - \bar{n}\text{-rank}\{f_i(y_i)\}\right) \end{aligned}$$

is sup-contractive for all n. By Corollary 3.5, x_n is asymptotically periodic with period P.

4. Finding the periodic limit

We now return to the max-type equation (2) and rank-type equation (5) and attempt to find closed-form solutions. The closed-form solution to the autonomous contractive rank-type equation was first given in [10], and we are able to reprove this result as a special case. However, we will see that finding a closed formula for the limit under periodic forcing is in general more difficult. We will need two lemmas about contractions on $\mathbb{R}$.

LEMMA 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction with fixed point r, then $x > r$ implies $f(x) < x$, and $x < r$ implies $f(x) > x$.*

Proof. Since $f(r) = r$ and f is a contraction, $|f(x) - r| < |x - r|$. If $x > r$, then

$$f(x) - r \leq |f(x) - r| < |x - r| = x - r,$$

so $f(x) < x$. If $x < r$, then

$$r - f(x) \leq |f(x) - r| < |x - r| = r - x,$$

so $f(x) > x$. □

LEMMA 4.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be contractions, and assume that r_1, r_2, r_3, r_4 satisfy $f(r_2) = r_1 < r_3$ and $g(r_1) = r_2 < r_4$. Then, either $f(r_4) < r_3$ or $g(r_3) < r_4$.*

Proof. Assume $g(r_3) \geq r_4$. Then,

$$r_4 - r_2 = r_4 - g(r_1) \leq g(r_3) - g(r_1) < r_3 - r_1.$$

The contractivity of f yields

$$f(r_4) - r_1 \leq |f(r_4) - r_1| = |f(r_4) - f(r_2)| < |r_4 - r_2| = r_4 - r_2 < r_3 - r_1,$$

which implies that $f(r_4) < r_3$. □

First, we can use Lemma 4.1 to find the explicit solution of the autonomous equation (4), where each f_i is a contraction with fixed point r_i. The following result, first proved in [10], shows that every initial condition converges to the constant solution k-rank $\{r_i\}$.

THEOREM 4.3. *Let* $f_i : \mathbb{R} \to \mathbb{R}$ *be a contraction with fixed point* r_i *for* $1 \leq i \leq M$ *and set* $x_n = k\text{-rank}\ \{f_i(x_{n-i})\}$. *Then,* $\lim_{n\to\infty} x_n = k\text{-rank}\{r_i\}$.

Proof. Note that this recurrence is autonomous and thus $P = 1$, so by Corollary 3.6 every initial condition converges to a unique constant (period 1) solution x^*. Thus, it remains only to show that $x^* = k\text{-rank}\{r_i\}$ is a fixed point of the recurrence. Let $\sigma : \{1, \ldots, M\} \to \{1, \ldots, M\}$ be a permutation such that

$$r_{\sigma(1)} \leq r_{\sigma(2)} \leq \cdots \leq r_{\sigma(M)}.$$

So, the constant solution should be $x^* = r_{\sigma(k)}$. Assume $x_1 = x_2 = \cdots = x_M = x^*$. Then,

$$\begin{aligned} x_{M+1} &= k\text{-rank}\{f_1(x^*), \ldots, f_{\sigma(k)}(x^*), \ldots, f_M(x^*)\} \\ &= k\text{-rank}\{f_1(x^*), \ldots, x^*, \ldots, f_M(x^*)\}. \end{aligned}$$

We want to show that $x_{M+1} = x^*$. Note that when $i < k$, we have $r_{\sigma(i)} \leq r_{\sigma(k)} = x^*$, so by Lemma 3.1 we have $f_{\sigma(i)}(x^*) \leq x^*$. Similarly, when $i > k$, we have $r_{\sigma(i)} \geq r_{\sigma(k)} = x^*$, so by Lemma 3.1, we have $f_{\sigma(i)}(x^*) \geq x^*$. Thus, we have

$$f_{\sigma(1)}(x^*), \ldots, f_{\sigma(k-1)}(x^*) \leq x^* \leq f_{\sigma(k+1)}(x^*), \ldots, f_{\sigma(M)}(x^*).$$

Thus, $x_{M+1} = x^*$, so x^* is a fixed point of the recurrence and therefore it is the unique limit for any initial condition. □

Theorem 4.3 gives the complete solution to the rank-type equation (5) in the period 1 case. We now turn to the case of period 2 forcing and restrict our attention to the max-type equation (2), for which it is possible to find a closed-form solution. This solution gives insight into the complexity of solutions to (2) when the forcing period is large. For $M = P = 2$, the equation (5) becomes the period 2 recurrence

$$\begin{aligned} x_{2i} &= \max\{f_1(x_{2i-1}), f_2(x_{2i-2})\} \\ x_{2i+1} &= \max\{g_1(x_{2i}), g_2(x_{2i-1})\}, \end{aligned} \tag{10}$$

where f_1, f_2, g_1 and g_2 are contractions. We can denote the fixed points

$$\begin{aligned} f_1(g_1(r_1)) = f_1(r_2) &= r_1 \\ g_1(f_1(r_2)) = g_1(r_1) &= r_2 \\ f_2(r_3) &= r_3 \\ g_2(r_4) &= r_4. \end{aligned}$$

THEOREM 4.4. *The period-2 orbit*

$$\begin{aligned} x_{2i} &= \max\{f_1(\max\{r_2, r_4\}), r_3\} \\ x_{2i+1} &= \max\{g_1(\max\{r_1, r_3\}), r_4\}, \end{aligned} \tag{11}$$

for $i \geq 0$ *is the attracting period-2 orbit of difference equation* (10).

Proof. Due to Corollary 3.6, it suffices to show that the formula (11) gives a period-2 orbit. First note the following table can be obtained easily.

$\max\{r_1, r_3\}$	$\max\{r_2, r_4\}$	x_{2i}	x_{2i+1}
r_1	r_2	r_1	r_2
r_3	r_4	$\max\{r_3, f_1(r_4)\}$	$\max\{r_4, g_1(r_3)\}$
r_1	r_4	$\max\{r_3, f_1(r_4)\}$	r_4
r_3	r_2	r_3	$\max\{r_4, g_1(r_3)\}$

For example, in the first row, we have:

$$x_{2i} = \max\{f_1(\max\{r_2, r_4\}), r_3\} = \max\{f_1(r_2), r_3\} = \max\{r_1, r_3\} = r_1.$$

The rest follows from similar simple logic. Now, we can make a new table considering all the possibilities for x_{2i}, x_{2i+1} and $x_{2i+2} = \max\{f_1(x_{2i+1}), f_2(x_{2i})\}$.

x_{2i}	x_{2i+1}	$f_1(x_{2i+1})$	$f_2(x_{2i})$	x_{2i+2}
r_1	r_2	r_1	$f_2(r_1)$	r_1
r_3	r_4	$f_1(r_4)$	r_3	r_3
$f_1(r_4)$	r_4	$f_1(r_4)$	$f_2(f_1(r_4))$	$f_1(r_4)$
r_3	$g_1(r_3)$	$f_1(g_1(r_3))$	r_3	r_3

Since columns three and four are clear, we must justify the final column using Lemma 4.1, (note that the final column is simply the max of columns three and four). In the first row, since $r_1 > r_3$ and r_3 is the fixed point of f_2, Lemma 4.1 implies that $f_2(r_1) < r_1$. In the second row, note that $r_3 = \max\{f_1(r_4), r_3\}$ so $r_3 > f_1(r_4)$. In the third row, note that $f_1(r_4) > r_3$ and since r_3 is the fixed point of f_2 by Lemma 4.1, we conclude that $f_2(f_1(r_4)) < f_1(r_4)$. Finally, in the fourth column, note that this combination only occurs when $r_3 > r_1$ and r_1 is the fixed point of $f_1 \circ g_1$ so by Lemma 4.1, we have $f_1(g_1(r_3)) < r_3$. Thus, the table shows that $x_{2i+2} = x_{2i}$.

Note that one combination, $x_{2i} = f_1(r_4)$ and $x_{2i+1} = g_1(r_3)$ is missing from the table. If this combination occurred, then $f_1(r_2) = r_1 < r_3 \leq f_1(r_4)$ and $g_1(r_1) = r_2 < r_4 \leq g_1(r_3)$, which contradicts Lemma 4.2. Thus, this combination is impossible, and the four rows of the table represent all possibilities.

We now construct an analogous table to show that $x_{2i+3} = x_{2i+1}$, which will complete the proof.

x_{2i+2}	x_{2i+1}	$g_1(x_{2i+2})$	$g_2(x_{2i+1})$	x_{2i+3}
r_1	r_2	r_2	$g_2(r_2)$	r_2
r_3	r_4	$g_1(r_3)$	r_4	r_4
$f_1(r_4)$	r_4	$g_1(f_1(r_4))$	r_4	r_4
r_3	$g_1(r_3)$	$g_1(r_3)$	$g_2(g_1(r_3))$	$g_1(r_3)$

In the first row, $r_2 > r_4$ so by Lemma 4.1, $g_2(r_2) < r_2$. In the second row, $r_4 = \max\{g_1(r_3), r_4\}$ so $r_4 > g_1(r_3)$. In the third row, $r_4 > r_2$ so by Lemma 4.1, $g_1(f_1(r_4) < r_4)$. In the final row, $g_1(r_3) > r_4$ so by Lemma 4.1, we have

$g_2(g_1(r_3)) < g_1(r_3)$. This completes the justification of this table, and finishes the proof. □

Example 4.5. We return to the equation from Example 1.5 with $M = P = 2$

$$\begin{aligned} x_{2i} &= \max\left\{A_{11}x_{2i-1}^{\alpha_1}, A_{21}x_{2i-2}^{\alpha_2}\right\} \\ x_{2i+1} &= \max\left\{A_{12}x_{2i}^{\alpha_1}, A_{22}x_{2i-1}^{\alpha_2}\right\}, \end{aligned}$$

where $A_{jk} > 0$ and $-1 < \alpha_1, \alpha_2 < 1$. We rewrite this equation by taking the natural log of each term (since ln is monotonic) to get

$$\begin{aligned} y_{2i} &= \max\{\ln A_{11} + \alpha_1 y_{2i-1}, \ln A_{21} + \alpha_2 y_{2i-2}\} \\ y_{2i+1} &= \max\{\ln A_{12} + \alpha_1 y_{2i}, \ln A_{22} + \alpha_2 y_{2i-1}\}. \end{aligned}$$

In terms of Theorem 4.4, we have

$$\begin{aligned} r_1 &= \frac{\ln A_{12} + \alpha_1 \ln A_{11}}{1 - \alpha_1^2} \\ r_2 &= \frac{\ln A_{11} + \alpha_1 \ln A_{12}}{1 - \alpha_1^2} \\ r_3 &= \frac{\ln A_{22}}{1 - \alpha_2} \\ r_4 &= \frac{\ln A_{21}}{1 - \alpha_2}. \end{aligned}$$

Then, the period-2 limit is defined in Theorem 4.4, and we can exponentiate to get the period-2 limit of x_n. Thus, we have

$$\begin{aligned} x_{2i} &= \max\left\{A_{12}A_{21}^{(\alpha_1/1-\alpha_2)}, A_{12}A_{11}^{\left(\alpha_1/1-\alpha_1^2\right)}A_{12}^{\left(\alpha_1^2/1-\alpha_1^2\right)}, A_{22}^{(1/1-\alpha_2)}\right\} \\ x_{2i+1} &= \max\left\{A_{11}A_{22}^{(\alpha_1/1-\alpha_2)}, A_{11}A_{12}^{\left(\alpha_1/1-\alpha_1^2\right)}A_{11}^{\left(\alpha_1^2/1-\alpha_1^2\right)}, A_{21}^{(1/1-\alpha_2)}\right\}, \end{aligned}$$

as the periodic limit. This solution is consistent with the autonomous case in Example 2.6 of [11]. Setting $A_1 = A_{11} = A_{12}$ and $A_2 = A_{21} = A_{22}$, the solution is found to be simply $x_n \to \max\left\{A_1^{(1/1-\alpha_1)}, A_2^{(1/1-\alpha_2)}\right\}$.

5. Discussion

We have shown in Corollary 3.6 that solutions of periodically forced rank-type difference equations are asymptotically periodic of the forcing period. The same is true of a class of more general equations called sup-contractive, according to Corollary 3.5. In some simple cases, we were able to identify explicit solutions, as in Theorems 4.3 and 4.4. The solutions appear to be significantly more complicated for larger period P and memory M than treated here, and explicit formulas for the solutions remain to be found.

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