Dimension Reduction in Spatiotemporal Dynamics

Tyrus Berry, Rob Cressman, Tim Sauer

Department of Mathematical Sciences
George Mason University
Fairfax VA 22030

Abstract

When a thin liquid crystal is driven by a sufficient A/C voltage, electroconvection produces complex spatiotemporal patterns. There is significant interest in quantifying this spatiotemporal complexity using measures of dimensionality and Lyapunov exponents, but these estimates are extremely difficult with experimental data because of the high dimensionality of the raw data. Simulations of models, such as Raleigh-Benard convection, indicate that there may be a low-dimensional representation of the process [1]. However, conventional techniques of dimensionality reductions such as the Karhunen-Loeve Decomposition have been unable to recover a low dimensional process even for low drivings levels [1, 2].

We have applied nonlinear dimensionality reduction techniques to significantly decrease the number of dimensions needed to represent a given proportion of the total spatiotemporal variance. By sampling representative sub-videos we construct a low-dimension state space and a vector field that represents the dynamics of all sub-videos simultaneously. These low-dimensional representations allow estimation of Lyapunov exponents from experimental data, and may lead to new models of spatiotemporal chaotic dynamics.

Thin Liquid Crystals

Applying an A/C voltage to a thin liquid crystal produces complex spatiotemporal patterns shown below. By varying the applied voltage we observe a continuous range of dynamic behaviors.

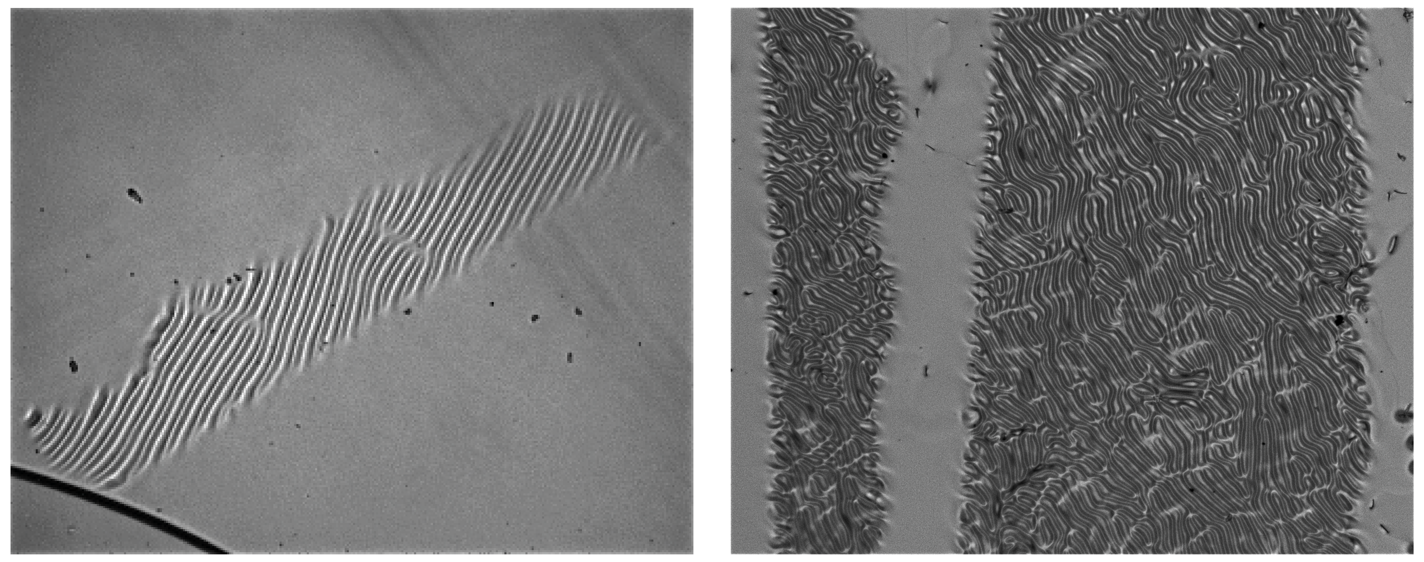


Figure 1: Two liquid crystals driven at low voltage (left) and high voltage (right).

Ridges and Defects

The two important features to note in the above images are the continuous ridges and the defects. At low driving voltages, the ridges move slowly across the image and only deviate slightly from their preferred orientation. The defects are discontinuities in the ridges and move more quickly and chaotically around the image. At the higher driving voltages, the ridges begin to have larger deviations from their preferred orientation and their movement becomes more complex. The concentration of defects becomes higher and the defects move very quickly and sporadically. Below we see enlarged images of the respective defects.

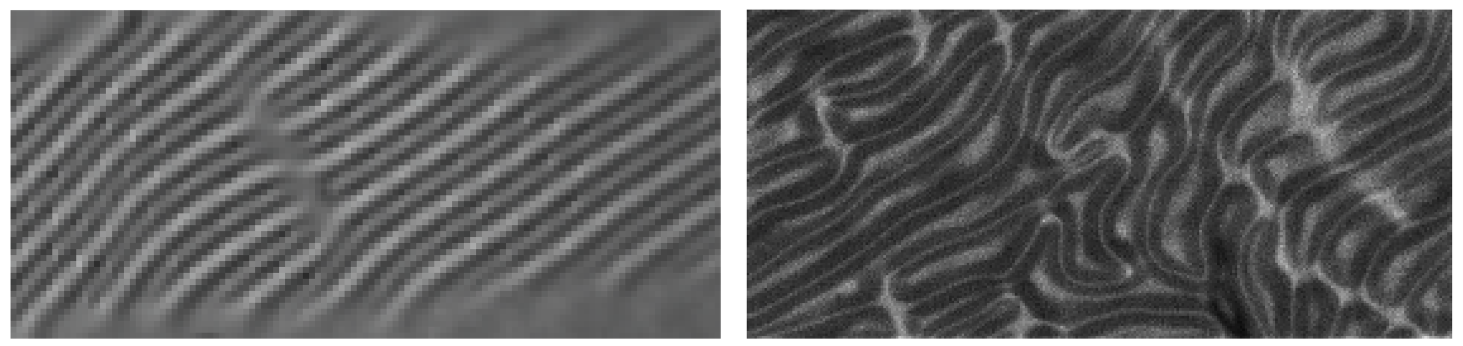


Figure 2: Enlarged image showing defects at low voltage (left) and high voltage (right).

Nonlinear Dimensionality Reduction with Isomap

Isomap [3] is a Nonlinear Dimensionality Reduction technique based on a modification of Multi-Dimensional Scaling (MDS) [4]. Isomap requires that the high-dimensional input data lies near a low-dimensional manifold. The idea is to construct low-dimensional coordinates by applying MDS to inter-point distances which are computed on the manifold. To find these distances, Isomap starts with a k-nearest-neighbor graph consisting of Euclidean distances. Then all inter-point distances are computed as the length of the shortest path in the graph. Finally, MDS is applied to the matrix of inter-point distances.

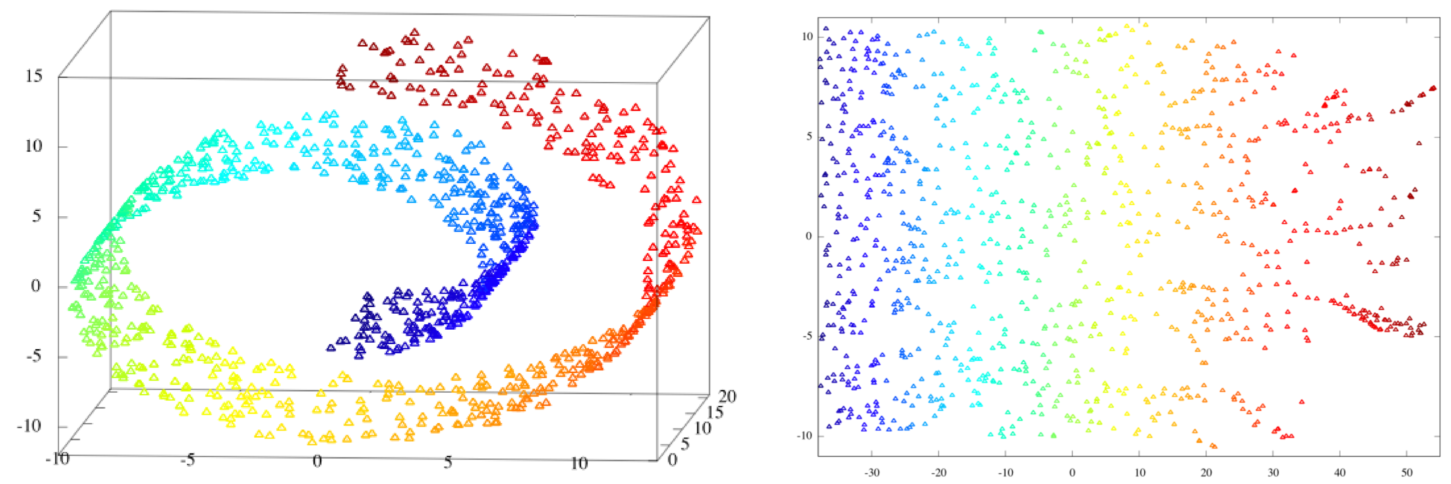
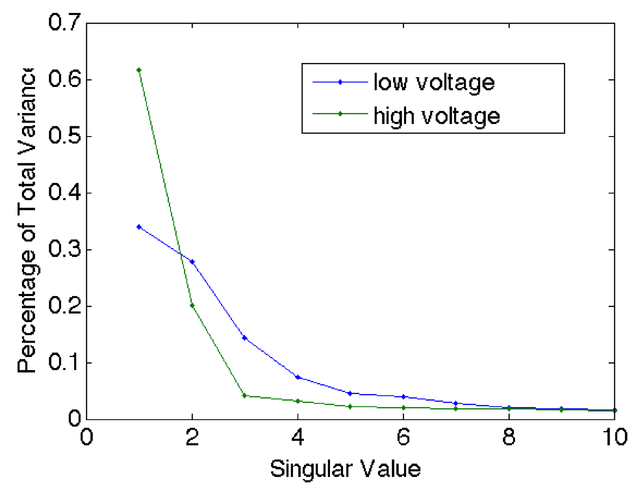


Figure 3: Isomap applied to a 2D manifold (left) produces the intrinsic coordinates (right)

DISO: Isomap for Spatiotemporal Dynamics

For dynamical data the state must incorporate spatial and temporal information. If the vector $\overline{x}$ represents an 8x8 sub-image, then the evolution of $\overline{x}(t)$ should be restricted to a manifold, which we assume has representation as a level set $F(\overline{x}, \overline{x}', ..., \overline{x}^{(n)}) = 0$. However, since the goal is to find a low dimensional representation of the dynamics, we are not interested in finding F explicitly. Instead we use time-delay coordinates to construct an equivalent manifold, $G(\overline{x}(t), \overline{x}(t-\tau_1), ..., \overline{x}(t-\tau_m)) = 0$. The assumption that G represents a low-dimensional manifold can be verified experimentally by observing the decay of the singular values from MDS [4]. The first ten singular values for this experiment are shown below.



Since nearby pixels and adjacent video frames are correlated, differences in these coordinates will typically be small. However, we can enhance these distances, and de-emphasize differences between distant pixels, by defining a new metric as $d(i,j) = \sqrt{(\overline{y}_i - \overline{y}_j)^T S (\overline{y}_i - \overline{y}_j)}$ where S is the correlation matrix of the coordinates of the vectors $\{\overline{y}_i\}$. By changing metric before applying the Isomap algorithm we observed faster singular value decays and qualitatively better decompositions.

Summary of DISO Algorithm

1. Choose a representative sample of sub-images.
2. For each sub-image form a state vector, $\overline{y}_i$ by appending several time delayed images.
3. Compute the covariance matrix, S, of the coordinates of the state vector.
4. Compute the initial distance matrix as $d(i,j) = \sqrt{(\overline{y}_i - \overline{y}_j)^T S (\overline{y}_i - \overline{y}_j)}$.
5. For each column of d set all distances to "inf" except for the k nearest neighbors.
6. Use the all-to-all version of Dijkstra's Algorithm to fill in "inf" values.
7. Find the Singular Value Decomposition of the resulting matrix of distances.
8. Use the Singular Values to determine the dimension of the manifold.
9. Find the reduced coordinates by projecting on the appropriate Singular Vectors.

The Reduced State Space

We can visualize the resulting low-dimensional state space by plotting the original sub-images according to their new coordinates (shown below). Note that in the right image, the x-axis captures information about how large the ridge in the sub-image is, while the y-axis captures information about the location of the ridge. In the left image the meaning of the coordinates is not as clear, and it is likely that the coordinates are capturing a complex spatiotemporal property which varies continuously in both space and time.

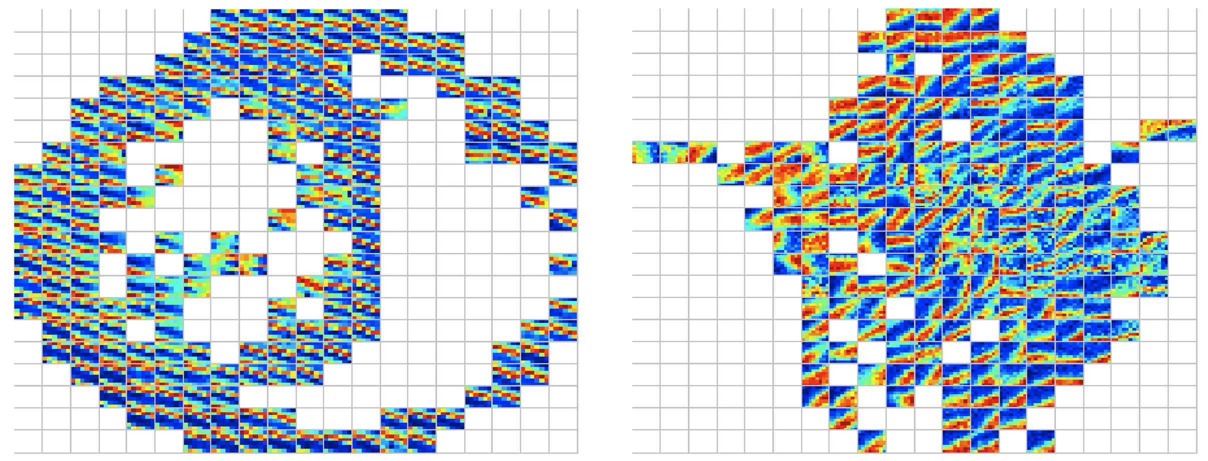


Figure 4: Sample subimages from the low voltage (left) and high voltage (right) crystals

Resulting Low Dimensional Dynamics

The images below show the results of each sub-image being projected into a reduced state space. The arrows indicated the trajectory from the previous time period to the current. The alignment of these trajectories shows that the low voltage spatiotemporal dynamics have a large deterministic component. The high voltage state space dynamics also shows some local alignment. This system may benefit from a higher frame rate.

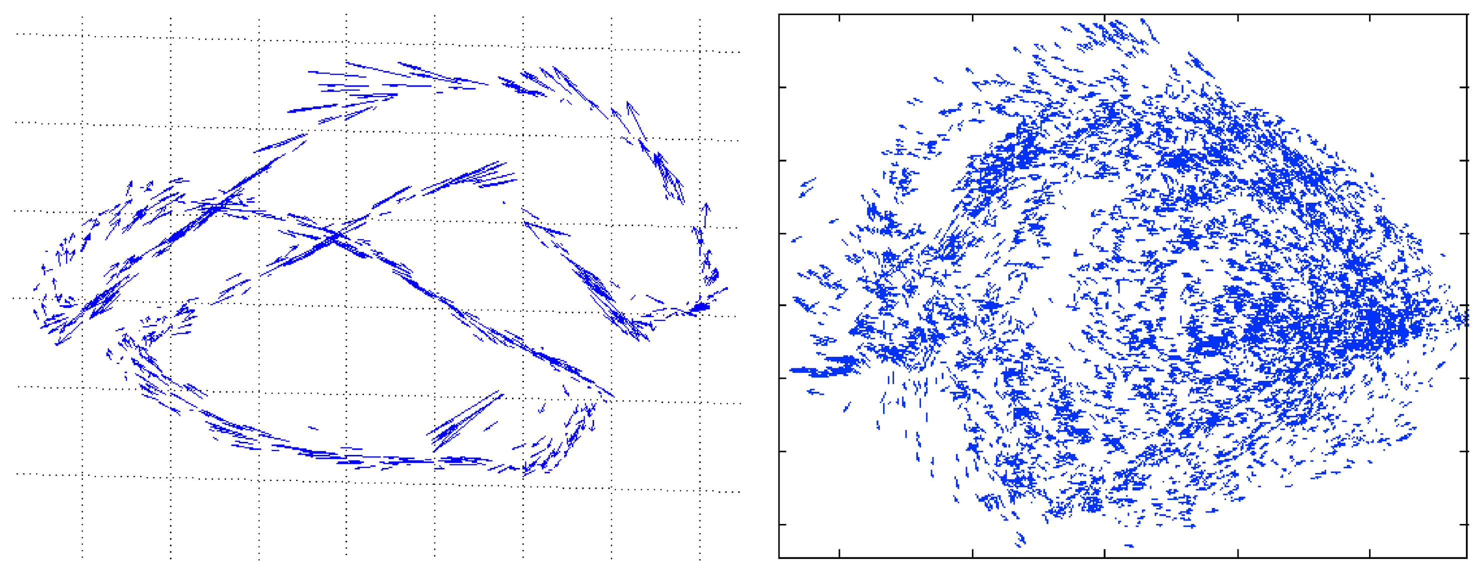


Figure 5: Low voltage dynamics (left) and high voltage dynamics (right)

The DISO algorithm successfully adapts the Isomap algorithm to spatiotemporal dynamics and reduces a 256 dimensional spatiotemporal state space to just three coordinates while preserving 85% of the original variance. Spatiotemporal dynamics, such as those exhibited by thin liquid crystals, are particularly difficult to analyze because of the high dimensionality of the data. By reducing the dimensionality it may now be possible to quantify the dynamics experimentally using Local Lyapunov Exponents or Lyapunov Dimension Density.

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[1] Egolf et al., *Mechanisms of extensive spatiotemporal chaos in Raleigh-Benard convection*, Nature **404** (2000), 733-736.

[2] Dangelmayr et al., *Diagnosis of spatiotemporal chaos in wave-envelopes of a nematic electroconvection pattern*, Physical Review E **79** (2009), 46215-46235.

[3] Tenenbaum et al., *A Global Geometric Framework for Nonlinear Dimensionality Reduction*, Science **290** (2000), 2319-2324.

[4] John A. Lee and Michel Verleysen, *Nonlinear Dimensionality Reduction*, Springer- Verlag New York, Inc., New York, New York, 2007.