Topology of diagonal arrangements and
flag enumerations of matroid base polytopes

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Abstract

This thesis provides results on combinatorial properties of diagonal arrangements and matroid base polytopes.

We study the topology of diagonal subspace arrangements in the first part. We prove that if a simplicial complex \( \Delta \) is shellable, then the intersection lattice \( L_\Delta \) for the corresponding diagonal arrangement \( A_\Delta \) is homotopy equivalent to a wedge of spheres. Furthermore, we describe precisely the spheres in the wedge, based on the data of shelling. As a consequence, we give the homotopy type and the homology of the singularity link (and hence the homology of the complement) of diagonal subspace arrangement \( A_\Delta \). Also, we give some examples of diagonal arrangements \( A \) where the complement \( M_A \) is \( K(\pi, 1) \), coming from rank 3 matroids.

In the second part, we study flag structures of matroid base polytopes. We describe faces of matroid base polytopes in terms of matroid data, and give conditions for hyperplane splits of matroid polytopes. Also, we apply this to the cd-index of a matroid base polytope of a rank 2 matroid.
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Chapter 1

Introduction

The objective of this work is to understand combinatorial properties of diagonal subspace arrangements and matroid base polytopes. More precisely, we study the topology of the intersection lattice of the diagonal subspace arrangement corresponding to a shellable simplicial complex. Also, we present some results about flags of faces in matroid base polytopes, and their cd-index.

This chapter provides informal summaries of Chapter 3 and 4. These summaries aim to be as non-technical as possible and precise definitions will be postponed until Chapter 2.

1.1 Topology of diagonal arrangements

In Chapter 3, we give the homotopy type of the intersection lattice of the diagonal arrangement corresponding to a shellable simplicial complex. More precisely, we show that this intersection lattice is homotopy equivalent to a wedge of spheres and describe the spheres in the wedge based on the data of shelling.

A simplicial complex on a vertex set \([n] = \{1, 2, \ldots, n\}\) is a collection of subsets of \([n]\) closed under taking subsets. The elements of a simplicial complex are called faces and maximal faces are called facets. A simplicial complex is shellable if there is a nice linear ordering, called a shelling, of facets (the formal definition will be given in Section 2.1). One of the nice properties of a shellable simplicial complex is that it is homotopy equivalent to a wedge of spheres. A shellable simplicial
complex $\Delta$ on $[5] = \{1, 2, \ldots, 5\}$ with a shelling $F_1 = 123$ (short for $\{1, 2, 3\}$), $F_2 = 234$, $F_3 = 35$, $F_4 = 45$ is shown in Figure 1.1.

Consider $\mathbb{R}^n$ with coordinates $u_1, u_2, \ldots, u_n$. A diagonal subspace is a linear subspace of $\mathbb{R}^n$ of the form $u_{i_1} = \cdots = u_{i_r}$ with $r \geq 2$. A diagonal arrangement $\mathcal{A}$ is a finite set of diagonal subspaces of $\mathbb{R}^n$. There is a one-to-one correspondence between diagonal arrangements of $\mathbb{R}^n$ and simplicial complexes $\Delta$ on $[n] = \{1, 2, \ldots, n\}$ with $\dim \Delta \leq n - 3$ (see Section 3.1).

The simplest example of diagonal arrangements is the thin diagonal, which consists of one line $u_1 = \cdots = u_n$. The corresponding simplicial complex is the simplicial complex on $[n]$ whose unique face is the empty set. Another well-known example of diagonal arrangements is the thick diagonal, which is the collection of all subspaces $u_i = u_j$ for $1 \leq i < j \leq n$. The thick diagonal corresponds to the simplicial complex on $[n]$ which consists of all subsets of $[n]$ having less than $n - 1$ elements.

Two important spaces associated with an arrangement $\mathcal{A}$ of linear subspaces in $\mathbb{R}^n$ are

$$\mathcal{M}_\mathcal{A} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{V}_\mathcal{A}^\circ = S^{n-1} \cap \bigcup_{H \in \mathcal{A}} H,$$

called the complement and the singularity link.

We are interested in the topology of $\mathcal{M}_\mathcal{A}$ and $\mathcal{V}_\mathcal{A}^\circ$ for a diagonal arrangement $\mathcal{A}$. In the mid 1980’s Goresky and MacPherson [20] found a formula for the Betti numbers of $\mathcal{M}_\mathcal{A}$, while the homotopy type of $\mathcal{V}_\mathcal{A}^\circ$ was computed by Ziegler and Živaljević [39] (see Section 2.3). The answers are phrased in terms of the lower intervals in the intersection lattice $L_\mathcal{A}$ of the subspace arrangement $\mathcal{A}$, that is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse
inclusion. For general subspace arrangements, these lower intervals in $L_A$ can have arbitrary homotopy type (see [39, Corollary 3.1]). When we say that a finite lattice has some topological properties, such as shellability and homotopy type, it means the order complex of its proper part has those properties. The formal definition of lattices and order complexes will be given in Section 2.2.

Our goal is to find a general sufficient condition for the intersection lattice $L_A$ of a diagonal arrangement $A$ to be well-behaved. Suggested by a homological calculation (Theorem 3.5.3 below), we will prove the following main result (see Section 3.4).

**Theorem 3.1.1.** Let $\Delta$ be a shellable simplicial complex. Then the intersection lattice $L_\Delta$ for the corresponding diagonal arrangement $A_\Delta$ is homotopy equivalent to a wedge of spheres.

The intersection lattice in Theorem 3.1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres (see Example 3.1.2).

Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. For the precise description, we introduce a notion of *shelling-trapped decomposition*, which is a decomposition of a subset of $[n]$ related to a shelling of $\Delta$, in Section 3.1.

Let $\Delta$ be a shellable simplicial complex on $[n]$ and $\bar{\sigma}$ be the complement of the intersection of all facets of $\Delta$. Then the wedge of spheres in Theorem 3.1.1 consists of $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p} |F_{ij}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition

$$D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$$

of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition $D$ of $\bar{\sigma}$ and a permutation $w$ of $[p-1]$, there exists a saturated chain $C_{D,w}$ (see Section 3.4) such that removing the simplices corresponding to these chains leaves a contractible simplicial complex. Note that the number and the dimension of spheres in a wedge are independent of the choice of shelling.
Figure 1.2 shows the intersection lattice of the diagonal arrangement $\mathcal{A}_\Delta$ corresponding to the simplicial complex $\Delta$ in Figure 1.1 and the order complex for its proper part. The chains $C_{D,w}$ and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines. See Example 3.4.13 for more details.

The homotopy type of the singularity link of $\mathcal{A}_\Delta$ for a shellable simplicial complex $\Delta$ is obtained from Theorem 3.1.1 and the result of Ziegler and Živaljević [39].

**Corollary 3.4.14.** Let $\Delta$ be a shellable simplicial complex on $[n]$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension

$$n + p(2 - n) + \sum_{j=1}^{p} |F_{ij}| - 2$$

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i1}), \ldots, (\bar{\sigma}_p, F_{ip})\}$ of some subset of $[n]$.

The homology version of this corollary (Theorem 3.5.3) can be proven without Theorem 3.1.1 by combining a result of Peeva, Reiner and Welker (Proposition 3.5.1) with results of Herzog, Reiner and Welker [22, Theorem 4, Theorem 9] along with the theory of Golod rings (see Section 3.5). It is what motivated us to
prove the stronger Corollary 3.4.14 and eventually Theorem 3.1.1.

1.2 Flag enumerations of matroid base polytopes

Chapter 4 is about flag structure and enumeration for matroid base polytopes. We associate a poset for each face of a matroid base polytope. We also give conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane and express the cd-index of matroid base polytopes of rank 2 matroids in terms of cd-indices of matroid base polytopes of corresponding indecomposable matroids.

A matroid is a combinatorial abstraction of vector configurations over any field, of hyperplane arrangements, and of graphs. For example, consider a collection of four vectors labeled 1, 2, 3, and 4 in $\mathbb{R}^2$ where vectors 1 and 2 are parallel. This collection forms a matroid, denoted $M_{2,1,1}$, on a ground set $[4] = \{1, 2, 3, 4\}$ and its vector configuration is shown in Figure 1.3(a). The rank of a matroid $M$, denoted $r(M)$, is the maximal number of independent vectors. For example, the matroid $M_{2,1,1}$ has rank 2. The precise definition and some properties of matroids will be given in Section 2.4.
For a matroid $M$ on a ground set $[n] = \{1, 2, \ldots, n\}$, a matroid base polytope $Q(M)$ is the polytope in $\mathbb{R}^n$ whose vertices are the incidence vectors of the bases of $M$. For the matroid $M_{2,1,1}$ on a ground set $[4] = \{1, 2, 3, 4\}$, its matroid base polytope is shown in Figure 1.3(b).

Since faces of a matroid base polytope $Q(M)$ are also matroid base polytopes, one can associate a matroid $M_\sigma$ on $[n]$ for each face $\sigma$ of $Q(M)$. Generalizing the characterization of facets of $Q(M)$ by Feichtner and Sturmfels [18, Proposition 2.6], we show that equivalence classes of factor-connected flags (defined in Section 4.2) of subsets of $[n]$ characterize faces of a matroid base polytope $Q(M)$ in Proposition 4.2.6. As a result, one can associate a poset for each face of $Q(M)$:

**Theorem 4.1.1.** Let $M$ be a matroid on a ground set $[n]$. For a face $\sigma$ of the matroid base polytope $Q(M)$, one can associate a poset $P_\sigma$ defined as follows:

(i) the elements of $P_\sigma$ are the connected components (defined in Section 4.2) of $M_\sigma$,

(ii) for distinct connected components $C_1$ and $C_2$ of $M_\sigma$, $C_1 \prec C_2$ if and only if $C_2 \subset S \subset [n]$ and $\sigma \subset H_S$ implies $C_1 \subset S$,

where $H_S$ is the hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = r(S)$, and $r(S)$ is the rank of $S$ (defined in Section 2.4).

If $\sigma$ is a vertex of a matroid base polytope, then this poset coincides with the poset defined by Billera, Jia and Reiner [4].

We found conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane.

**Theorem 4.3.1.** Let $M$ be a rank $r$ matroid on $[n]$ and $H$ be a hyperplane in $\mathbb{R}^n$ given by $\sum_{e \in S} x_e = k$. Then $H$ decomposes $Q(M)$ into two matroid base polytopes if and only if

(i) $r(S) \geq k$ and $r(S^c) \geq r - k$,

(ii) if $I_1$ and $I_2$ are $k$-element independent subsets of $S$ such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$,
where $M|_S$ is the restriction of $M$ on $S$ and $M/S$ is the contraction of $M$ on $S$ (see Section 2.4 for the precise definition).

The cd-index $\Psi(Q)$ of a polytope $Q$, a polynomial in the noncommutative variables $c$ and $d$, is an invariant that compactly encodes a large amount of enumerative information about flags of faces in $Q$. The cd-index will be defined in Section 4.4 for more general posets, namely Eulerian posets. Generalizing the formula of the cd-index of a prism and a pyramid of a polytope given by Ehrenborg and Readdy [16], we give the following theorem.

**Theorem 4.4.2.** Let $Q$ be a polytope in $\mathbb{R}^n$ and $H$ be a hyperplane in $\mathbb{R}^n$. Then the following identity holds:

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\hat{Q}) \cdot c - \sum_\sigma \Psi(\hat{\sigma}) \cdot d \cdot \Psi(\hat{Q}/\hat{\sigma}),$$

where the sum is over all proper faces $\sigma$ of $Q$ intersecting both open halfspaces obtained by $H$ nontrivially (notations will be defined in Section 2.4 and Section 4.4).

We apply Theorem 4.3.1 and Theorem 4.4.2 to express the cd-index of a matroid base polytope for a rank 2 matroid in terms of cd-indices of matroid base polytopes of corresponding indecomposable matroids (Proposition 4.5.1).
Chapter 2

Preliminaries

This chapter contains five sections, devoted respectively to simplicial complexes, posets and their order complexes, subspace arrangements, matroids, and convex polytopes. In each of these sections, we provide the basic background materials which will be used throughout this thesis. Readers familiar with topics of a section may want to skip that section. We refer [28] for topological notions and [13] for notions on commutative algebra.

2.1 Simplicial complexes

This section contains basic information about shellable simplicial complexes.

An (abstract) simplicial complex $\Delta$ on a finite vertex set $V$ is a collection of subsets of $V$ satisfying

$\tau \subset \sigma \in \Delta$ implies $\tau \in \Delta$.

For $\sigma \in \Delta$, define $\dim \sigma = |\sigma| - 1$ and $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. The elements of $\Delta$ are called faces and the maximal faces are called facets. A simplicial complex $\Delta$ is pure if each facet has the same dimension.

The geometric realization $|\Delta|$ of the simplicial complex $\Delta$ is an embedding in which each vertex maps to a point and each (abstract) simplex maps to the (geometric) simplex spanned by the images of its vertices. Abusing notation, we sometimes identify a simplicial complex with its geometric realization.

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For a simplicial complex $\Delta$ on the vertex set $V$, the \textit{(Alexander) dual complex} $\Delta^*$ of $\Delta$ is defined to be

$$\Delta^* = \{ \bar{\sigma} \subset V : \sigma = V - \bar{\sigma} \not\in \Delta \}.$$  

The \textit{simplicial join} of two simplicial complex $\Delta$ and $\Delta'$ with disjoint vertex sets $V$ and $V'$ respectively, denoted by $\Delta \ast \Delta'$, is the simplicial complex on $V \cup V'$ with faces $\sigma \cup \tau$ where $\sigma$ is a face of $\Delta$ and $\tau$ is a face of $\Delta'$.

A simplicial complex is \textit{shellable} if its facets can be arranged in linear order $F_1, F_2, \ldots, F_q$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} 2F_i) \cap 2F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, q$, where $2F = \{ G \mid G \subseteq F \}$. Such an ordering of facets is called a \textit{shelling order} or \textit{shelling}. For further background on this notion, see [8], [9].

There are several equivalent definitions of shellability. The following restatement of shellability is often useful.

\textbf{Lemma 2.1.1.} ([8, Lemma 2.3]) A linear order $F_1, F_2, \ldots, F_q$ of facets of a simplicial complex is a shelling if and only if for every $F_i$ and $F_k$ with $F_i < F_k$, there is a facet $F_j < F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k \ll F_k$, where $G \ll F$ means that $G$ has codimension 1 in $F$.

\textbf{Example 2.1.2.} The simplicial complexes in Figure 2.1(a) and (c) are shellable with shelling orders $F_1 = 12, F_2 = 13, F_3 = 23$ and $F_1 = 123, F_2 = 234, F_3 = 35, F_4 = 45$, respectively. On the other hand, since the intersection of two facets in Figure 2.1(b) is empty, i.e., has codimension two in any facet, the simplicial complex in Figure 2.1(b) is not shellable.

The \textit{link} of a face $\sigma$ of a simplicial complex $\Delta$ is

$$\text{link}_\Delta \sigma (= \text{link} \sigma) = \{ \tau \in \Delta \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta \}.$$  

Björner and Wachs [9] show that the shellability is inherited by all links of faces in a simplicial complex.

\textbf{Proposition 2.1.3.} If $\Delta$ is shellable, then so is $\text{link}_\Delta \sigma$ for all faces $\sigma \in \Delta$, using the induced order on facets of $\text{link}_\Delta \sigma$. 

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Example 2.1.4. For the simplicial complex in Figure 2.1(d), one can see that link \{3\} is not shellable since it is the simplicial complex in Figure 2.1(b) after relabeling. Thus Proposition 2.1.3 shows that the simplicial complex in Figure 2.1(d) is not shellable.

It is well-known that a pure \(d\)-dimensional shellable simplicial complex has the homotopy type of a wedge of \(d\)-spheres. Björner and Wachs [8] generalize this result to the nonpure case, i.e., a nonpure shellable simplicial complex has the homotopy type of a wedge of spheres, however these spheres need not be equidimensional.
2.2 Posets and their order complexes

This section contains background information about posets and their order complexes. The poset terminology and notations used in this thesis are standard and generally agree with [35], where proofs of the basic results in this section can be found.

A partially ordered set (or a poset) \( P \) is a set together with a reflexive, anti-symmetric, transitive relation \( \leq \) (or \( \leq_P \) when there is a possibility of confusion). A subposet \( Q \) is a subset of \( P \) with the partial order induced on \( Q \). If \( x \leq y \), the closed interval \([x, y]\) is \( \{z \in P : x \leq z \leq y\} \) and the open interval \((x, y)\) is \( \{z \in P : x < z < y\} \).

If \( x, y \in P \), we say \( y \) covers \( x \) and write \( x \lessdot y \) if \( x < y \) and if no element \( z \in P \) satisfies \( x < z < y \). The Hasse diagram of a finite poset \( P \) is the graph whose vertices are the elements of \( P \), whose edges are the cover relations, and \( y \) is drawn “above” \( x \) if \( x < y \).

If there exists a unique minimal element in \( P \), it is denoted \( \hat{0} \). Similarly, if \( P \) has a unique maximal element, it is denoted \( \hat{1} \). A poset \( P \) is bounded if it has both \( \hat{0} \) and \( \hat{1} \). A chain is a poset in which any two elements are comparable. A subset of a poset \( P \) is called a chain if it is a chain when regarded as a subposet of \( P \). The chain \( C \) of \( P \) is called saturated if there does not exist \( z \in P - C \) such that \( x < z < y \) for some \( x, y \in C \) and such that \( C \cup \{z\} \) is also a chain. A chain is maximal if it is not properly contained in any other chain. The length \( l(C) \) of a finite chain \( C \) is defined by \( l(C) = |C| - 1 \). A poset \( P \) is graded (or pure) of rank \( n \) if every maximal chain of \( P \) has the same length \( n \). In this case, there is a unique rank function \( \rho \) such that \( \rho(x) = 0 \) if \( x \) is minimal, and \( \rho(y) = \rho(x) + 1 \) if \( y \) covers \( x \).

For \( x, y \in P \), if there is a unique minimal element in \( \{z \in P : z \geq x, z \geq y\} \), it is called the join of \( x \) and \( y \) and denoted \( x \lor y \). Similarly, if there is a unique maximal element in \( \{z \in P : z \leq x, z \leq y\} \), it is called the meet of \( x \) and \( y \) and denoted \( x \land y \). A lattice is a poset \( L \) for which every pair of elements \( x \) and \( y \) has \( x \lor y \) and \( x \land y \).

The order complex \( \Delta(P) \) of a poset \( P \) is the simplicial complex whose vertices
are the elements of $P$ and whose faces are the chains of $P$. Figure 2.2 is an example of a poset and its order complex. Note both $P$ and $\Delta(P)$ in Figure 2.2 are nonpure.

For the order complex $\Delta((x,y))$ of an open interval $(x,y)$, we will use the notation $\Delta(x,y)$. When we say that a finite lattice $L$ has some topological properties, such as purity, shellability and homotopy type, it means the order complex of $\overline{L} = L - \{\hat{0}, \hat{1}\}$ has those properties.

### 2.3 Subspace arrangements

In this section, we give basic information about subspace arrangements.

Let $\mathbb{K}$ be a field. A linear subspace of $\mathbb{K}^n$ is a subspace of $\mathbb{K}^n$ containing $0 \in \mathbb{K}^n$, and an affine subspace of $\mathbb{K}^n$ is a translate of a linear space of $\mathbb{K}^n$. A subspace arrangement $\mathcal{A}$ in $\mathbb{K}^n$ is a finite collection of affine proper subspaces of $\mathbb{K}^n$. A subspace arrangement is called central if all subspaces are linear spaces. Throughout this thesis, we will take $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Consider $\mathbb{K}^n$ with coordinates $u_1, \ldots, u_n$. A diagonal subspace $U_{i_1 \ldots i_r}$ is a linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$ with $r \geq 2$. A diagonal arrangement (or a hypergraph arrangement) is a subspace arrangements consisting of diagonal subspaces.

The simplest example of diagonal arrangements is the thin diagonal, which
consists of one line \( u_1 = \cdots = u_n \). Another well-known example of diagonal arrangements is the *thick diagonal*, which is the collection of all subspaces \( u_i = u_j \) for \( 1 \leq i < j \leq n \). This arrangement is also known as the *Braid arrangement* \( B_n \) and is well-studied (see [36] and Section 3.2 below). These two arrangements are particular cases of *\( k \)-equal arrangements* \( A_{n,k} \). The \( k \)-equal arrangement \( A_{n,k} \) in \( \mathbb{K}^n \) is the arrangement of all subspaces \( u_{i_1} = \cdots = u_{i_k} \) defined by setting \( k \) coordinates equal.

Two important spaces associated with a central arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \) are

\[
\mathcal{M}_\mathcal{A} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{V}_\mathcal{A} = S^{n-1} \cap \bigcup_{H \in \mathcal{A}} H,
\]

called the *complement* and the *singularity link*. Note that \( \mathcal{M}_\mathcal{A} \) and \( \mathcal{V}_\mathcal{A} \) are related by Alexander duality as follows:

\[
H^i(\mathcal{M}_\mathcal{A}; \mathbb{F}) = H_{n-2-i}(\mathcal{V}_\mathcal{A}; \mathbb{F}) \quad (\mathbb{F} \text{ is any field}).
\]

In the mid 1980’s Goresky and MacPherson [20] found a formula for the Betti numbers of \( \mathcal{M}_\mathcal{A} \), i.e., the ranks of the singular cohomology groups \( H^i(\mathcal{M}_\mathcal{A}) \). The answer is phrased in terms of the lower intervals in the intersection lattice of the subspace arrangement \( \mathcal{A} \). The *intersection lattice* \( L_\mathcal{A} \) of a subspace arrangement \( \mathcal{A} \) is the collection of all nonempty intersections of subspaces of \( \mathcal{A} \) ordered by reverse inclusion.

**Theorem 2.3.1.** Let \( \mathcal{A} \) be a subspace arrangement in \( \mathbb{R}^n \). Then

\[
\tilde{\beta}^i(\mathcal{M}_\mathcal{A}) = \sum_{x \in L_\mathcal{A} - \{\hat{0}\}} \tilde{\beta}_{\text{codim}(x)-2-i}(\hat{0}, x),
\]

where the Betti number \( \tilde{\beta}_d(P) \) of a poset \( P \) is the rank of the \( d \)-dimensional reduced simplicial homology group of the order complex \( \Delta(P) \).

The homotopy type of \( \mathcal{V}_\mathcal{A} \) is computed by Ziegler and Živaljević [39].

**Theorem 2.3.2.** For every central subspace arrangement \( \mathcal{A} \) in \( \mathbb{R}^n \),

\[
\mathcal{V}_\mathcal{A} \simeq \bigvee_{x \in L_\mathcal{A} - \{\hat{0}\}} (\Delta(\hat{0}, x) \ast S^{\text{dim}(x)-1}),
\]

where \( \bigvee \) denotes wedge of spaces, “\( \ast \)” denotes join of spaces, and “\( \simeq \)” denotes homotopy equivalence.
These two results show that the topologies of $M_A$ and $V^n_A$ reduce to topology of (lower) intervals in $L_A$. For general subspace arrangements, these lower intervals in $L_A$ can have arbitrary homotopy type (see [39, Corollary 3.1]).

2.4 Matroids

There are many equivalent ways to define a matroid (see [29], [32], [38] for more details). In this section, we provide two axiom systems for matroids.

A **matroid** $M$ is an ordered pair $(E, I)$ where $E = E(M)$ is a finite set and $I = I(M)$ is a collection of subsets of $E$ satisfying the following three conditions:

(I1) $\emptyset \in I$,

(I2) If $I_1 \in I$ and $I_2 \subset I_1$, then $I_2 \in I$,

(I3) If $I_1, I_2 \in I$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in I$.

If $M$ is the matroid $(E, I)$, $M$ is called a matroid on $E$. The elements of $I$ are called the **independent sets** of $M$. A maximal independent subset of $M$ is called a base of $M$. All bases of $M$ have the same size and the rank of $M$ is the size of a base of $M$.

The next theorem shows that a matroid can be defined using bases.

**Theorem 2.4.1** ([29, Corollary 1.2.5]). A set $B$ of subsets of $E$ is the collection of bases of a matroid on $E$ if and only if it satisfies the following conditions:

(B1) $B \neq \emptyset$,

(B2) If $B_1, B_2 \in B$ and $x \in B_1 - B_2$, then there is an element $y$ of $B_2$ such that $(B_1 - \{x\}) \cup \{y\} \in B$.

For a matroid $M$ on $E$ and $e \in E$, $e$ is an **isthmus** (or coloop) if $e$ is contained in every base in $B(M)$. If $e$ is not an isthmus, the **deletion** $M\setminus e$ is a matroid on $E - \{e\}$ having independent sets

$$I(M\setminus e) = \{I \in I(M) : e \notin I\}$$
or bases

\[ \mathcal{B}(M \setminus e) = \{ B \in \mathcal{B}(M) : e \notin B \}. \]

For a subset \( S = \{e_1, \ldots, e_k\} \) of \( E \), one can define a deletion

\[ M \setminus S := ((M \setminus e_1) \setminus e_2) \ldots \setminus e_k \]

and the restriction

\[ M|_S := M \setminus (E - S). \]

The rank \( r(S) \) of \( S \) is the rank of \( M|_S \).

Given a matroid \( M \) on \( E \) and \( e \in E \), one says that \( e \) is a loop if \( e \) lies in none of the bases in \( \mathcal{B}(M) \). If \( e \) is not a loop, the contraction \( M/e \) is a matroid on \( E - \{e\} \) having independent sets

\[ \mathcal{I}(M/e) = \{ I - \{e\} : I \in \mathcal{I}(M), e \in I \} \]

or bases

\[ \mathcal{B}(M/e) = \{ B - \{e\} : B \in \mathcal{B}(M), e \in B \}. \]

For a subset \( S = \{e_1, \ldots, e_k\} \) of \( E \), the contraction on \( S \) is defined by

\[ M/S := ((M/e_1)/e_2) \ldots /e_k. \]

For a matroid \( M \) on \( E \), its dual matroid \( M^\perp \) is a matroid on \( E \) having bases

\[ \mathcal{B}(M^\perp) = \{ E - B : B \in \mathcal{B}(M) \}. \]

### 2.5 Convex polytopes

This section provides basic information about convex polytopes.

Recall that an affine subspace of \( \mathbb{R}^n \) is a translate of linear subspace of \( \mathbb{R}^n \). The dimension of an affine subspace is the dimension of the corresponding linear subspace. The affine hull of a set \( S \subset \mathbb{R}^n \) is the intersection of all affine subspaces containing \( S \).

A set \( C \subset \mathbb{R}^n \) is convex if for any two points \( x, y \in C \) it also contains the straight line segment with endpoints \( x \) and \( y \). For any \( S \subset \mathbb{R}^n \), the convex
The convex hull of $S$, denoted $\text{conv}(S)$, is the intersection of all convex sets containing $S$. A hyperplane in $\mathbb{R}^n$ is the solution set of an equality $l(x) = c$ for some linear function $l$ on $\mathbb{R}^n$ and some vector $c$ of $\mathbb{R}^n$, and the corresponding closed halfspace is the solution set of an inequality $l(x) \geq c$. A convex polytope is the convex hull of a finite set of points. Equivalently, a convex polytope is a bounded set which is an intersection of finitely many closed halfspaces. The dimension of a convex polytope is the dimension of its affine hull.

A hyperplane $l(x) = c$ is a supporting hyperplane of a convex polytope $Q$ if $l(x) \geq c$ for every point in $Q$. A face of $Q$ is any intersection of $Q$ with some supporting hyperplane. In particular, $\emptyset$ and $Q$ itself are always faces of $Q$. Any face of $Q$ is itself a convex polytope. The faces of dimension $0, 1, \text{ and } \text{dim}(Q) - 1$ are called vertices, edges, and facets, respectively. The face lattice of $Q$ is a set of all faces of $Q$ partially ordered by inclusion.

If $Q$ is a $d$-dimensional polytope and $x_0$ is a point outside the affine hull of $Q$ (for this we embed $Q$ into $\mathbb{R}^n$ for some $n > d$), then the convex hull

$$\text{Pyr}(Q) := \text{conv}(Q \cup \{x_0\})$$

is a $(d + 1)$-dimensional polytope called the pyramid over $Q$. Similarly, we construct the bipyramid over $Q$

$$\text{Bipy}(Q) := \text{conv}(Q \cup \{x_+, x_\}$$

by choosing two points $x_+$ and $x_-$ outside the affine hull of $Q$ such that an interior point of the segment $[x_+, x_-]$ is an interior point of $Q$.

For polytopes $Q_1 \subset \mathbb{R}^n$ and $Q_2 \subset \mathbb{R}^m$,

$$Q_1 \times Q_2 := \{(x, y) : x \in Q_1, y \in Q_2\}$$

is a polytope of dimension $\text{dim} Q_1 + \text{dim} Q_2$ and is called the product of $Q_1$ and $Q_2$. In particular, the Prism over a polytope $Q$ is the product of $Q$ with a line segment,

$$\text{Prism}(Q) := Q \times \Delta_1,$$

where $\Delta_1$ is a line segment (i.e., 1-dimensional polytope).
Let \( v \) be a vertex of a polytope \( Q \) and let \( l(x) = c \) be a supporting hyperplane of \( Q \) defining \( v \). The \textit{vertex figure} \( Q/v \) of \( v \) is defined by

\[
Q/v = Q \cap \{l(x) = c + \delta\}
\]

where \( \delta \) is an arbitrary small positive number. For a face \( \sigma \) of \( Q \), the \textit{face figure} \( Q/\sigma \) of \( \sigma \) is defined by

\[
Q/\sigma = \cdots ((Q/\sigma_0)/\sigma_1) \cdots)/\sigma_k
\]

where \( \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k = \sigma \) is a maximal chain with \( \dim \sigma_i = i \). For faces \( \sigma \) and \( \tau \) of \( Q \) with \( \sigma \subset \tau \), the face lattice of the face figure \( \tau/\sigma \) is the interval \([\sigma, \tau]\).
Chapter 3

Topology of Diagonal Arrangements

3.1 Main results

For a simplicial complex $\Delta$ on a vertex set $[n]$ with $\dim \Delta \leq n - 3$, one can associate a diagonal arrangement $\mathcal{A}_\Delta$ as follows. For a facet $F$ of $\Delta$, let $U_F$ be the diagonal subspace $u_{i_1} = \cdots = u_{i_r}$ where $\overline{F} = [n] - F = \{i_1, \ldots, i_r\}$. Define

$$\mathcal{A}_\Delta = \{U_F | F \text{ is a facet of } \Delta\}.$$

For each diagonal arrangement $\mathcal{A}$, one can find a simplicial complex $\Delta$ such that $\mathcal{A} = \mathcal{A}_\Delta$.

Our goal is to find a general sufficient condition for the intersection lattice $L_{\mathcal{A}}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved. Björner and Welker [10] show that $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres, where $\mathcal{A}_{n,k}$ is the $k$-equal arrangement in $\mathbb{R}^n$ (see Section 3.3). More generally, Kozlov [25] shows that $L_{\mathcal{A}}$ is shellable if $\mathcal{A}$ satisfies certain technical conditions (see Section 3.3). Suggested by a homological calculation (Theorem 3.5.3 below), we will prove the following main result, capturing the homotopy type assertion from [25] (see Section 3.4).

**Theorem 3.1.1.** Let $\Delta$ be a shellable simplicial complex. Then the intersection lattice $L_{\Delta}$ for the diagonal arrangement $\mathcal{A}_{\Delta}$ is homotopy equivalent to a wedge of spheres.
Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. Let $\Delta$ be a simplicial complex on $[n]$ with a shelling order $F_1, \ldots, F_q$ on its facets. Let $\sigma$ be the intersection of all facets, and $\bar{\sigma}$ its complement. Let $G_1 = F_1$ and for each $i \geq 2$ let $G_i$ be the face of $F_i$ obtained by intersecting the walls of $F_i$ that lie in the subcomplex generated by $F_1, \ldots, F_{i-1}$, where a wall of $F_i$ is a codimension 1 face of $F_i$. An (unordered) shelling-trapped decomposition of $\bar{\sigma}$ (over $\Delta$) is defined to be a family $\{(\bar{\sigma}, F_{i_1}), \ldots, (\bar{\sigma}, F_{i_p})\}$ such that $\{\bar{\sigma}, \ldots, \bar{\sigma}\}$ is a decomposition of $\bar{\sigma}$ as a disjoint union

$$\bar{\sigma} = \bigsqcup_{j=1}^p \bar{\sigma}_j$$

and $F_{i_1}, \ldots, F_{i_p}$ are facets of $\Delta$ such that $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all $j$. Then the wedge of spheres in Theorem 3.1.1 consists of $(p - 1)!$ copies of spheres of dimension

$$p(2 - n) + \sum_{j=1}^p |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}, F_{i_1}), \ldots, (\bar{\sigma}, F_{i_p})\}$ of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition $D$ of $\bar{\sigma}$ and a permutation $w$ of $[p - 1]$, there exists a saturated chain $C_{D,w}$ (see Section 3.4) such that removing the simplices corresponding to these chains leaves a contractible simplicial complex.

The following example shows that the intersection lattice in Theorem 3.1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres.

**Example 3.1.2.** Let $\Delta$ be a simplicial complex on $[8] = \{1, 2, \ldots, 8\}$ with a shelling 123456, 127, 237, 137, 458, 568, 468. Then $\Delta(U_{78}, \hat{1})$ is a disjoint union of two circles, hence is not shellable. Therefore, the intersection lattice $L_\Delta$ for the diagonal arrangement $A_\Delta$ is also not shellable. The intersection lattice $L_\Delta$ is shown in Figure 3.1 (thick lines represent the open interval $(U_{78}, \hat{1})$). In Figures, the subspace $U_{i_1 \ldots i_r}$ is labeled by $i_1 \ldots i_r$. Also note that a facet $F$ of $\Delta$ corresponds to the subspace $U_{[n]-F}$. For example, the facet 127 corresponds to $U_{34568}$.

The next example shows that there are nonshellable simplicial complexes whose intersection lattices are shellable.

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Figure 3.1: Nonshellable intersection lattice $L_\Delta$ for shellable $\Delta$ ($U_{i_1 \ldots i_r}$ is labeled by $i_1 \ldots i_r$.)

**Example 3.1.3.** Let $\Delta$ be a simplicial complex on $\{1, 2, 3, 4\}$ whose facets are 12 and 34. Then $\Delta$ is not shellable as we have seen in Example 2.1.2. But the order complex of $L_\Delta$ consists of two vertices, hence is shellable.

This chapter is organized as follows: Section 3.2 contains applications to group cohomology. In Section 3.3, we give Kozlov’s result and show that its homotopy type consequence is a special case of Theorem 3.1.1. Also, we give a new proof of Björner and Welker’s result using Theorem 3.1.1. Section 3.4 gives a proof of Theorem 3.1.1, and we deduce the homotopy type of the singularity link of $A_\Delta$ for a shellable simplicial complex $\Delta$. In Section 3.5, we give the homology of the singularity link (and hence the homology of the complement) of a diagonal arrangement $A_\Delta$ for a shellable simplicial complex $\Delta$ without using Theorem 3.1.1. In Section 3.6, we give some examples in which $M_A$ are $K(\pi, 1)$, coming from matroids.
3.2 Motivation from group cohomology

In this section, we give an application of topology of diagonal arrangements to group cohomology. Recall that an Eilenberg-MacLane space (or a $K(\pi, n)$-space) is a connected cell complex with all homotopy groups except the $n$-th homotopy group being trivial and the $n$-th homotopy group isomorphic to $\pi$.

Let the CW complex $X$ be a $K(\pi, 1)$-space. Let $\tilde{X}$ be its universal cover, i.e., the unique simply connected cover. Let $p : \tilde{X} \to X$ be the covering map. Then we have

(i) \( \tilde{X} \) is also a CW complex.

(ii) $\pi_1(X) = \pi$ acts freely on $\tilde{X}$. Explicitly, the open cells of $\tilde{X}$ lying over an open cell $\sigma$ of $X$ are simply the connected components of $p^{-1}\sigma$. These cells are permuted freely and transitively by $\pi$, and each is mapped homeomorphically onto $\sigma$ under $p$. Thus $C_\ast(\tilde{X})$ is a complex of free $\mathbb{Z}\pi$-modules with one basis element for each cell of $X$.

(iii) $\pi_1(\tilde{X}) = 1$ and $\pi_i(\tilde{X}) = \pi_i(X) = 0$ for $i > 1$. Hence $\tilde{X}$ is contractible.

Thus the augmented cellular complex of $\tilde{X}$

\[
\cdots \to C_2(\tilde{X}) \to C_1(\tilde{X}) \to C_0(\tilde{X}) \to C_{-1}(\tilde{X}) \to 0
\]

\[
\cdots \to (\mathbb{Z}\pi)^{\beta_2} \to (\mathbb{Z}\pi)^{\beta_1} \to (\mathbb{Z}\pi)^{\beta_0} \to \mathbb{Z} \to 0,
\]

where $\beta_i$ is the number of $i$-dimensional cells of $X$, is exact. Hence removing $C_{-1}(\tilde{X})$ (i.e., taking the usual cellular complex of $\tilde{X}$) gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ (see [11, Proposition 4.1]).

Tensoring with $\mathbb{Z}$ over $\mathbb{Z}\pi$ and taking homology gives $\text{Tor}^\mathbb{Z}_n(\mathbb{Z}, \mathbb{Z})$. On the other hand, the same evaluation gives $H_\ast(X; \mathbb{Z})$ since

\[
C_i(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z} = (\mathbb{Z}\pi)^{\beta_i} \otimes_{\mathbb{Z}\pi} \mathbb{Z} = \mathbb{Z}^{\beta_i} = C_i(X; \mathbb{Z}) \quad \text{for all } i.
\]

Hence

\[
\text{Tor}^\mathbb{Z}_n(\mathbb{Z}, \mathbb{Z}) = H_n(X; \mathbb{Z}).
\]

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Similarly, one can get

$$\text{Ext}^n_{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z}) = H^n(X; \mathbb{Z})$$

by applying the functor $\text{Hom}_{\mathbb{Z}\pi}(-, \mathbb{Z})$ and taking cohomology. Therefore, one can evaluate $\text{Tor}$ and $\text{Ext}$ of $\mathbb{Z}$ in terms of the integral (co)homology of the $K(\pi, 1)$-space $X$.

Now, we give two examples of diagonal arrangement whose complement is a $K(\pi, 1)$-space.

Let $B(n)$ be the braid group on $n$ strands. Figure 3.2(a) shows a braid on three strands. There is a natural surjection $B(n) \to \mathfrak{S}_n$ which sends each braid to the permutation of its ends. The image of the braid in Figure 3.2(a) is the 3-cycle $(1, 3, 2)$. The kernel of this map is called the pure braid group $PB(n)$. The corresponding pure braids have the property that each strand returns to its point of origin. Figure 3.2(b) shows a pure braid on three strands.

Consider a pure braid on $n$ strands as a set of $n$ non-intersecting arcs in $\mathbb{R}^2 \times [0, 1]$ such that

(i) arcs connect $(1, 0, 0), \ldots, (n, 0, 0)$ with $(1, 0, 1), \ldots, (n, 0, 1)$ in this order, and

(ii) arcs descend monotonically.
Define a path in $\mathbb{C}^n$ by
\[(a_1(t) + ib_1(t), \ldots, a_n(t) + ib_n(t)), \quad 0 \leq t \leq 1,
\]
where $(a_j(t), b_j(t), t)$ is on the arc connecting $(j, 0, 0)$ and $(j, 0, 1)$ for all $1 \leq j \leq n$.

Since $a_j(t) + ib_j(t) \neq a_k(t) + ib_k(t)$ whenever $j \neq k$, and $a_j(0) = a_j(1) = j$ and $b_j(0) = b_j(1) = 0$ for all $1 \leq j \leq n$, we get a closed path in $\mathcal{M}_{B_n}$ with the base point $(1, 2, \ldots, n)$.

By using this correspondence, one can show the second part of the following theorem given by Fadell and Neuwirth [17].

**Theorem 3.2.1.** Let $\mathcal{B}_n$ be the braid arrangement in $\mathbb{C}^n$. Then

(i) $\mathcal{M}_{B_n}$ is $K(\pi, 1)$ space, and

(ii) its fundamental group is isomorphic to the pure braid group $PB(n)$.

Khovanov [23] gives a real counterpart of Theorem 3.2.1. For that result, we need to define the *twin group*. Let $\mathcal{F}_n$ be the topological space of configurations of $n$ continuous arcs in the strip $\mathbb{R} \times [0, 1]$ such that

(i) arcs connect points $(1,1), \ldots, (n,1)$ with $(1,0), \ldots, (n,0)$ in some order,

(ii) arcs descend monotonically, and

(iii) no three arcs have a common point.

Two elements $a, b \in \mathcal{F}_n$ are multiplied by putting one on top of the other and squeezing the interval $[0, 2]$ to $[0, 1]$. Define a *twin* on $n$ arcs to be a connected component of the space $\mathcal{F}_n$. Then twins form a group and we call it the *twin group on $n$ arcs*. There is a natural surjection of the twin group to the symmetric group $\mathfrak{S}_n$ given by sending each twin to the permutation of its ends. The kernel of this surjection is called the *pure twin group on $n$ arcs*. An example of a twin and a pure twin on four arcs is shown in Figure 3.3.

For each pure twin on $n$ arcs, we can define the closed path in $\mathbb{R}^n$ based on $(1,2,\ldots,n)$ given by
\[(a_1(t), \ldots, a_n(t)), \quad 0 \leq t \leq 1\]
where \((a_i(t), t)\) lies on the arc that connects \((i, 0)\) and \((i, 1)\). Since no three arcs have a common point, this closed path lies in \(\mathcal{M}_{\mathcal{A}_{n,3}}\). This explains the second part of the following theorem given by Khovanov \[23\].

\textbf{Theorem 3.2.2.} Let \(\mathcal{A}_{n,3}\) be the 3-equal arrangement in \(\mathbb{R}^n\). Then

(i) \(\mathcal{M}_{\mathcal{A}_{n,3}}\) is a \(K(\pi, 1)\) space, and

(ii) the fundamental group of \(\mathcal{M}_{\mathcal{A}_{n,3}}\) is isomorphic to the pure twin group on \(n\) arcs.

\section{3.3 Special cases that were known}

In this section, we give Kozlov’s theorem and show how its consequence for homotopy type follows from Theorem 3.1.1. Also, we give a new proof of Björner and Welker’s theorem about the intersection lattice of the \(k\)-equal arrangements using Theorem 3.1.1.

Kozlov \[25\] shows that \(\mathcal{A}_\Delta\) has a shellable intersection lattice if \(\Delta\) satisfies some conditions. This class includes \(k\)-equal arrangements and all other diagonal arrangements for which the intersection lattice was proved shellable.
Theorem 3.3.1. ([25, Corollary 3.2]) Consider a partition of
\[ [n] = E_1 \sqcup \cdots \sqcup E_r \]
such that \( \max E_i < \min E_{i+1} \) for \( i = 1, \ldots, r - 1 \). Let
\[ f : \{1, 2, \ldots, r\} \to \{2, 3, \ldots\} \]
be a nondecreasing map. Let \( \Delta \) be a simplicial complex on \([n]\) such that \( F \) is a facet of \( \Delta \) if and only if
\begin{enumerate}[(i)]  
\item \( |E_i - F| \leq 1 \) for \( i = 1, \ldots, r \);
\item if \( \min F \in E_i \) then \( |F| = n - f(i) \).
\end{enumerate}
Then the intersection lattice for \( \mathcal{A}_\Delta \) is shellable. In particular, \( L_{\mathcal{A}_\Delta} \) has the homotopy type of a wedge of spheres.

Proposition 3.3.2. \( \Delta \) in Theorem 3.3.1 is shellable.

Proof. We claim that a shelling order for \( \Delta \) is \( F_1, F_2, \ldots, F_q \) such that the words \( w_1, w_2, \ldots, w_q \) are in lexicographic order, where \( w_i \) is the increasing array of elements in \( F_i \). Let \( F_s, F_t \) be two facets of \( \Delta \) with \( 1 \leq s < t \leq q \). Then \( w_s \prec_{\text{lex}} w_t \). Let \( m \) be the first number in \([r]\) such that \( E_m - F_s \neq E_m - F_t \). Construct the word \( w \) as follows:
\begin{enumerate}[(i)]  
\item \( w \cap E_i = w_s \cap E_i \) for \( i = 1, \ldots, m \);
\item for \( i = m + 1, \ldots, q \),
\[ w \cap E_i = \begin{cases} w_t \cap E_i & \text{if } |w \cap (\bigcup_{j=1}^{i} E_j)| \leq f(l); \\ \emptyset & \text{otherwise}, \end{cases} \]
\end{enumerate}
where \( \min w_s \in E_l \).

Note that the length of \( w \) is \( f(l) \) and \( w \prec_{\text{lex}} w_t \). Let \( F \) be a set of all elements which do not appear in \( w \). Since \( F \) satisfies the two conditions in Theorem 3.3.1, \( F \) is a facet of \( \Delta \). Since \( F \cap F_t = F_t - (E_m - F_s) \) and \( E_m - F_s \) is a subset of \( F_t \) of size 1, \( F \cap F_t \) has codimension 1 in \( F_t \). Also \( F_s \cap F_t \subseteq F \cap F_t \). Hence \( F_1, F_2, \ldots, F_q \) is a shelling by Lemma 2.1.1. \( \Box \)
Table 3.1: Facets of \( \Delta \) and corresponding words

Example 3.3.3. Consider the partition of

\[
[7] = \{1\} \sqcup \{2, 3\} \sqcup \{4\} \sqcup \{5, 6, 7\}
\]

and the function \( f \) given by \( f(1) = 2, f(2) = 3, f(3) = 4, \) and \( f(4) = 5 \). Then the facets of the simplicial complex that satisfy the conditions of Corollary 3.3.1 and the corresponding words are given in Table 3.1. Thus the ordering 34567, 24567, 23567, 23467, 23457, 1367, 1357, 1356, 1267, 1257 and 1256 is a shelling for this simplicial complex.

One can also use Theorem 3.1.1 to recover the following theorem of Björner and Welker [10].

Theorem 3.3.4. The order complex of the intersection lattice \( L_{\mathcal{A}_{n,k}} \) has the homotopy type of a wedge of spheres consisting of

\[
(p - 1)! \sum_{0 \leq i_0 \leq i_1 \leq \cdots \leq i_p = n - pk} \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}
\]

copies of \((n-3-p(k-2))\)-dimensional spheres for \(1 \leq p \leq \lfloor \frac{n}{k} \rfloor\).

Proof. It is clear that \( \mathcal{A}_{n,k} = \mathcal{A}_{\Delta_{n,n-k}} \), where \( \Delta_{n,n-k} \) is a simplicial complex on \([n]\) whose facets are all \( n-k \) subsets of \([n]\). Here, \( \sigma = \emptyset, \ \bar{\sigma} = [n] \). By ordering the elements of each facet in increasing order, the lexicographic order of facets of \( \Delta_{n,n-k} \) gives a shelling. Also, one can see that the facet of the form

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</table>


\( F_i = \{1, 2, \ldots, m, a_{m+1}, \ldots, a_{n-k}\}, \) where \( m + 1 < a_{m+1} < \cdots < a_{n-k}, \) has \( G_i = \{1, \ldots, m\}. \) Thus, \( G_i \subseteq \sigma_i \subseteq F_i \) implies \( \overline{F_i} \subseteq \overline{\sigma_i} \subseteq \overline{G_i} = \{m+1, \ldots, n\}. \) Note that \( \min \overline{\sigma} = \min \overline{F} = m+1 \) and \( \overline{F} \) has \( k \) elements. Thus, in any shelling-trapped decomposition \([n] = \bigsqcup_{i=1}^{p} \overline{\sigma}_i \), one has \( p \leq \lfloor \frac{n}{k} \rfloor \).

Let \( 1 \leq p \leq \lfloor \frac{n}{k} \rfloor \) and \( 0 = i_0 \leq i_1 \leq \cdots \leq i_p = n - pk \). We will construct a shelling-trapped family \( \{ (\overline{\sigma}_1, F_{i_1}), \ldots, (\overline{\sigma}_p, F_{i_p}) \} \) as in Theorem 3.1.1. Because \( F_{i_1} < \cdots < F_{i_p} \), we have \( \min \overline{\sigma}_1 > \cdots > \min \overline{\sigma}_p \). In particular, \( 1 \in F_{i_p} \subseteq \overline{\sigma}_p \). Thus there are \( \binom{n-1}{k-1} \) ways to pick \( F_{i_p} \) (equivalently, \( F_{i_p} \)). Now suppose that we have chosen \( F_{i_p}, \ldots, F_{i_p-j+1} \). We pick \( F_{i_p-j} \) so that \( \min \overline{F}_{i_p-j} = \min \overline{\sigma}_{i_p-j} \) is the \( i_j + 1 \)st element in \([n] - (F_{i_p} \cup \cdots \cup F_{i_p-j+1}) \). Then we have \( \binom{n-jk-i_j-1}{k-1} \) ways to choose \( F_{i_p-j} \). For each \( j = 1, \ldots, p \), there are \( i_j - i_{j-1} \) elements in

\[
[n] - (F_{i_p} \cup \cdots \cup F_{i_p-j+1})
\]

which are strictly between \( \min \overline{F}_{i_p-j+1} \) and \( \min \overline{F}_{i_p-j} \) and they must be contained in one of \( \overline{\sigma}_p, \ldots, \overline{\sigma}_{i_p-j+1} \) (i.e., there are \( j_{i_j} - i_{j-1} \) choices). Therefore there are

\[
\prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} \prod_{j=1}^{p} j_{i_j-i_{j-1}} = \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}
\]

shelling-trapped families. By Theorem 3.1.1, each of those families contributes \( (p-1)! \) copies of spheres of dimension

\[
p(2-n) + \sum_{j=1}^{p} (n-k) + n - 3 = n - 3 - p(k-2).
\]

### 3.4 Proof of main theorem

Theorem 3.1.1 will be deduced from a more general statement about homotopy types of lower intervals in \( L_A \), Theorem 3.4.1 below. Throughout this section, we assume that \( \Delta \) is a simplicial complex on \([n] \) with \( \dim \Delta \leq n - 3 \).
Theorem 3.4.1. Let $F_1, \ldots, F_q$ be a shelling of $\Delta$ and $U_{\bar{\sigma}}$ a subspace in $L_{\Delta}$ for some subset $\bar{\sigma}$ of $[n]$. Then $\Delta(\hat{0}, U_{\bar{\sigma}})$ is homotopy equivalent to a wedge of spheres, consisting of $(p - 1)!$ copies of spheres of dimension

$$\delta(D) := p(2 - n) + \sum_{j=1}^{p} |F_{ij}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$.

Moreover, for each such shelling-trapped decomposition $D$ and each permutation $w$ of $[p - 1]$, one can construct a saturated chain $C_{D,w}$ (see Section 3.4.1 below), such that if one removes the corresponding $\delta(D)$-dimensional simplices for all pairs $(D, w)$, the remaining simplicial complex $\hat{\Delta}(\hat{0}, U_{\bar{\sigma}})$ is contractible.

To prove this result, we begin with some preparatory lemmas.

First of all, one can characterize exactly which subspaces lie in $L_{\Delta}$ when $\Delta$ is shellable. Recall that for $\bar{\sigma} = \{i_1, \ldots, i_r\} \subseteq [n]$, we denote by $U_{\bar{\sigma}}$ the linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$. We also use the notation $U_{\bar{\sigma}_1 / \cdots / \bar{\sigma}_p}$ to denote $U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}$ for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$.

A simplicial complex is called gallery-connected if any pair $F, F'$ of facets are connected by a path

$$F = F_0, F_1, \ldots, F_{r-1}, F_r = F'$$

of facets such that $F_i \cap F_{i-1}$ has codimension 1 in $F_i$ for $i = 1, \ldots, r$. Since it is known that Cohen-Macaulay simplicial complexes are gallery-connected, shellable simplicial complexes are gallery-connected.

Lemma 3.4.2. (i) Given any simplicial complex $\Delta$ on $[n]$, every subspace $H$ in $L_{\Delta}$ has the form

$$H = U_{\bar{\sigma}_1 / \cdots / \bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$ such that $\sigma_i$ can be expressed as an intersection of facets of $\Delta$ for $i = 1, 2, \ldots, p$.

(ii) Conversely, when $\Delta$ is gallery-connected, every subspace $H$ of $\mathbb{R}^n$ that has the above form lies in $L_{\Delta}$.
Proof. To see (i), note that every subspace \( H \) in \( L_\Delta \) has the form

\[
H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}
\]

for pairwise disjoint subsets \( \bar{\sigma}_1, \ldots, \bar{\sigma}_p \) of \([n]\). Since \( H = \bigvee_{F \in \mathcal{F}} U_F \) for some family \( \mathcal{F} \) of facets of \( \Delta \),

\[
U_{\bar{\sigma}_j} = \bigvee_{F \in \mathcal{F}_j} U_F
\]

for some subfamily \( \mathcal{F}_j \) of \( \mathcal{F} \) for all \( j = 1, \ldots, p \). Therefore

\[
\sigma_j = \bigcap_{F \in \mathcal{F}_j} F
\]

for \( j = 1, \ldots, p \).

For (ii), suppose that \( H \) has the form

\[
H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}
\]

for pairwise disjoint subsets \( \bar{\sigma}_1, \ldots, \bar{\sigma}_p \) of \([n]\) such that \( \sigma_i \) can be expressed as an intersection of facets of \( \Delta \) for \( i = 1, 2, \ldots, p \). It is enough to show the case when \( H = U_\sigma \). Since gallery-connectedness is inherited by all links of faces in a simplicial complex, we may assume \( \sigma = \bigcap_{F \in \mathcal{F}} F \), where \( \mathcal{F} \) is the set of all facets of \( \Delta \), without loss of generality. Then \( \bar{\sigma} = \bigcup_{F \in \mathcal{F}} \bar{F} \).

We claim that the simplicial complex \( \Gamma \) whose facets are \( \{ \bar{F} \mid F \in \mathcal{F} \} \) is connected. Since \( \dim \Delta \leq n - 3 \), every facet of \( \Gamma \) has at least two elements. Let \( \bar{F}, \bar{F}' \) be two facets of \( \Gamma \) with \( F < F' \). Since \( \Delta \) is gallery-connected, there is \( F = F_1, F_2, \ldots, F_k = F' \) such that \( F_i \cap F_{i-1} \) has codimension 1 in \( F_i \) for all \( i = 2, \ldots, k \). Thus \( \bar{F}_i \) and \( \bar{F}_{i-1} \) share at least one vertex for all \( i = 2, \ldots, k \). This implies that \( \bar{F} \) and \( \bar{F}' \) are connected. Hence \( \Gamma \) is connected.

Therefore \( \bar{\sigma} = \bigcup_{F \in \mathcal{F}} \bar{F} = \bigvee_{F \in \mathcal{F}} \bar{F} \). \( \square \)

The next example shows that the conclusion of Lemma \ref{lem:gallery-connectedness}(ii) can fail when \( \Delta \) is not assumed to be gallery-connected.

**Example 3.4.3.** Let \( \Delta \) be a simplicial complex with two facets 123 and 345. Then \( \Delta \) is not gallery-connected. Since \( L_\Delta \) has only three subspaces \( U_{12}, U_{45} \) and \( U_{12/45} \), it does not have the subspace \( U_{1245} \), even though \( \overline{1245} = 3 \) is an intersection of facets 123 and 345 of \( \Delta \). Thus the conclusion of Lemma \ref{lem:gallery-connectedness}(ii) fails for \( \Delta \).
The following easy lemma shows that every lower interval $[\hat{0}, H]$ can be written as a product of lower intervals of the form $[\hat{0}, U_\bar{\sigma}]$.

**Lemma 3.4.4.** Let $H \in L_\Delta$ be a subspace of the form

$$H = U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$. Then

$$[\hat{0}, H] \cong [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

In particular,

$$|\Delta(\hat{0}, H)| \simeq |\Delta(\hat{0}, U_{\bar{\sigma}_1})| \ast \cdots \ast |\Delta(\hat{0}, U_{\bar{\sigma}_p})| \ast \mathcal{S}^{p-2},$$

where “$\ast$” denotes join of spaces.

**Proof.** We will use induction on $p$. If $K \leq H$ in $L_\Delta$, then $K$ has the form $K' \cap K''$ where $K' \leq U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_{p-1}}$ and $K'' \leq U_{\bar{\sigma}_p}$. Thus $[\hat{0}, H]$ is isomorphic to $[\hat{0}, U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_{p-1}}] \times [\hat{0}, U_{\bar{\sigma}_p}]$ as posets via the map $K' \cap K'' \mapsto (K', K'')$. By the induction hypothesis, the first result is obtained. The second result is obtained from [37, Theorem 5.1].

The next lemma, whose proof is completely straightforward and omitted, shows that the lower interval $[\hat{0}, U_\bar{\sigma}]$ is isomorphic to the intersection lattice for the diagonal arrangement corresponding to link$_\Delta \bar{\sigma}$.

**Lemma 3.4.5.** Let $U_\bar{\sigma}$ be a subspace in $L_\Delta$ for some face $\bar{\sigma}$ of $\Delta$. Then the lower interval $[\hat{0}, U_\bar{\sigma}]$ is isomorphic to the intersection lattice of the diagonal arrangement $A_{\text{link}_\Delta(\bar{\sigma})}$ corresponding to link$_\Delta(\bar{\sigma})$ on the vertex set $\bar{\sigma}$.

The following lemma shows that upper intervals in $L_\Delta$ are at least still homotopy equivalent to the intersection lattice of a diagonal arrangement.

**Lemma 3.4.6.** Let $U_\bar{\sigma}$ be a subspace in $L_\Delta$ for some face $\bar{\sigma}$ of $\Delta$. Then the upper interval $[U_\bar{\sigma}, \hat{1}]$ is homotopy equivalent to the intersection lattice of the diagonal arrangement $A_{\Delta_\bar{\sigma}}$ corresponding to the simplicial complex $\Delta_\bar{\sigma}$ on the vertex set $\bar{\sigma} \cup \{v\}$ whose facets are obtained in the following way:
(A) If $F \cap \sigma$ is maximal among
\[
\{F \cap \sigma \mid F \text{ is a facet of } \Delta \text{ with } \sigma \not\subseteq F \text{ and } F \cup \sigma \neq [n]\},
\]
then $\hat{F} = F \cap \sigma$ is a facet of $\Delta_\sigma$.

(B) If a facet $F$ of $\Delta$ satisfies $F \cup \sigma = [n]$, then $\hat{F} = (F \cap \sigma) \cup \{v\}$ is a facet of $\Delta_\sigma$.

Proof. We apply a standard crosscut/closure lemma ([7, Theorem 10.8]) saying that a finite lattice $L$ is homotopy equivalent to the sublattice consisting of the joins of subsets of its atoms. By the closure relation on $[U_\sigma, \hat{1}]$ which sends a subspace to the intersection of all subspaces that lie weakly below it and cover $U_\sigma$ in $[U_\sigma, \hat{1}]$, one can see that $[U_\sigma, \hat{1}]$ is homotopy equivalent to the sublattice $L_\sigma$ generated by the subspaces of $[U_\sigma, \hat{1}]$ that cover $U_\sigma$. By using the map $\psi$ defined by
\[
\psi(U_\mathfrak{r}) = \begin{cases} U_{(\mathfrak{r} - \sigma)} \cup \{v\}, & \text{if } \bar{\sigma} \cap \bar{\mathfrak{r}} \neq \emptyset, \\ U_\mathfrak{r}, & \text{otherwise}, \end{cases}
\]
one can see that $L_\sigma$ is isomorphic to the intersection lattice $L_{\Delta_\sigma}$ for a simplicial complex $\Delta_\sigma$ on the vertex set $\sigma \cup \{v\}$. The facets of $\Delta_\sigma$ correspond the subspaces that cover $U_\sigma$, giving the claimed characterization of facets of $\Delta_\sigma$. \qed

Example 3.4.7. Let $\Delta$ be a simplicial complex in Figure 2.1(c), i.e., a simplicial complex on $\{1, 2, 3, 4, 5\}$ with facets 123, 234, 35, 45 and let $\sigma = 123$. Then $\Delta_\sigma$ is a simplicial complex on $\{1, 2, 3, v\}$ and its facets are 23 and $v$. The intersection lattices $L_\Delta$ and $L_{\Delta_\sigma}$ are shown in Figure 3.4 and it is easy to see that the order complex for $\bar{L}_{\Delta_\sigma}$ is homotopy equivalent to the order complex for the interval $(U_{45}, \hat{1})$ in $L_\Delta$. Note that the thick lines in Figure 3.4(a) represent the closed interval $[U_{45}, \hat{1}]$ in $L_\Delta$.

In general, the simplicial complex $\Delta_\sigma$ of Lemma 3.4.6 is not shellable, even though $\Delta$ is shellable (see Example 3.1.2). However, the next lemma shows that $\Delta_\sigma$ is shellable if $\sigma$ is the last facet in the shelling order.
Lemma 3.4.8. Let $\Delta$ be a shellable simplicial complex. If $F$ is the last facet in a shelling order of $\Delta$, then $\Delta_F$ is also shellable. Moreover, if $\tilde{F}_i$ is a facet of $\Delta_F$ of type (B) (defined in Lemma 3.4.6), then $\tilde{G}_i = G_i \cap F$.

Proof. One can check that the following gives a shelling order on the facets of $\Delta_F$. First list the facets of type (A) according to the order of their earliest corresponding facet of $\Delta$, followed by the facets of type (B) according to the order of their corresponding facet of $\Delta$.

To see the second assertion, let $\tilde{F}_i$ be a facet of $\Delta_F$ of type (B), i.e.,

$$\tilde{F}_i = (F_i \cap F) \cup \{v\}$$

for some facet $F_i$ of $\Delta$ such that $F \cup F_i = [n]$. Then $\tilde{G}_i = G_i \cap F$ follows from the observations that $F_i \cap F$ is an old wall of $\tilde{F}_i$, and all other old walls of $\tilde{F}_i$ are $\tilde{F}_k \cap \tilde{F}_i$ for some facets $\tilde{F}_k$ of $\Delta_F$ of type (B) such that $F_k \cap F_i$ is an old wall of $F_i$.

Example 3.4.9. Let $\Delta$ be a shellable simplicial complex on $\{1, 2, \ldots, 7\}$ with a shelling $12367, 12346, 13467, 34567, 13457, 14567, 12345$ and let $F = 12345$. Then $\Delta_F$ is a simplicial complex on $\{1, 2, 3, 4, 5, v\}$ and its facets are $123v, 1234, 134v, 345v, 1345, 145v$. Since $1234, 1345$ are facets of $\Delta_F$ of type (A) and $123v$, $134v,$
134v, 345v, 145v are facets of $\Delta_F$ of type (B), the ordering 1234, 1345, 123v, 134v, 345v, 145v is a shelling of $\Delta_F$.

We next construct the saturated chains appearing in the statement of Theorem 3.4.1.

3.4.1 Constructing the chains $C_{D,w}$

Let $\Delta$ be shellable and let $U_\sigma$ be a subspace in $L_\Delta$. Let

$$D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$$

be a shelling-trapped decomposition of $\bar{\sigma}$ and let $w$ be a permutation on $[p - 1]$. We construct a chain $C_{D,w}$ in $[\hat{0}, U_\bar{\sigma}]$ as follows:

(i) By Lemma 3.4.2, the interval $[\hat{0}, U_\sigma]$ contains $U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}$ and the interval $[U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}, U_\sigma]$ is isomorphic to the set partition lattice $\Pi_p$. It is well known that the order complex of $\Pi_p = \Pi_p - \{\hat{0}, \hat{1}\}$ is homotopy equivalent to a wedge of $(p - 1)!$ spheres of dimension $p - 3$ and there is a saturated chain $C_w$ in $\Pi_p$ for each permutation $w$ of $[p - 1]$ such that removing $\{C_w = C_w - \{\hat{0}, \hat{1}\}|w \in S_{p-1}\}$ from the order complex of $\Pi_p$ gives a contractible subcomplex (see [5, Example 2.9]). Identify $U_{\bar{\sigma}_1}, \ldots, U_{\bar{\sigma}_p}$ with 1, \ldots, $p$ in this order and take the saturated chain $\tilde{C}_w$ in $[U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}, U_\sigma]$ which corresponds to the chain $C_w$ in $\Pi_p$.

(ii) By Lemma 3.4.4,

$$[\hat{0}, U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}] \cong [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

Since $\Delta$ is shellable and $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all $j$, one can see that $[\hat{0}, U_{\sigma_j}]$ has a subinterval $[U_{\bar{\sigma}_1}, U_{\sigma_j}]$ which is isomorphic to the boolean algebra of the set of order $|\bar{\sigma}_j| - |\bar{F}_{i_j}|$. Thus

$$[U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}, U_{\bar{\sigma}_1/\ldots/\bar{\sigma}_p}]$$

is isomorphic to

$$[U_{\bar{\sigma}_1}, U_{\sigma_1}] \times \cdots \times [U_{\bar{\sigma}_p}, U_{\sigma_p}]$$

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and hence is isomorphic to the boolean algebra of the set of order
\[\sum_{j=1}^{p} (|\overline{\sigma}_j| - |F_{ij}|)\,.

Take any saturated chain \(\tilde{C}\) in
\([U_{\mathcal{T}_1/\ldots/\mathcal{T}_{ip}}, U_{\overline{\sigma}_1/\ldots/\sigma_p}]\).

(iii) Define a saturated chain \(C_{D, w}\) by
\[\hat{0} \prec U_{\mathcal{T}_p} \prec U_{\mathcal{T}_p/\mathcal{T}_{ip-1}} \prec \cdots \prec U_{\mathcal{T}_p/\ldots/\mathcal{T}_{i_1}}\]
followed by the chains \(\tilde{C}\) and \(\tilde{C}_w\) (where \(\prec\) means the covering relation in \(L_\Delta\)).

Note that the length of the chain \(\overline{C}_{D, w} = C_{D, w} - \{\hat{0}, U_a\}\) is
\[l(\overline{C}_{D, w}) = p(2 - n) + \sum_{j=1}^{p} |F_{ij}| + |\overline{\sigma}| - 3.\]

**Example 3.4.10.** Let \(\Delta\) be the shellable simplicial complex in Example 3.4.9. Then one can see that
\[D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}\]
is a shelling-trapped decomposition of \(\{1, 2, 3, 4, 5, 6, 7\}\). Let \(w\) be the permutation in \(S_2\) with \(w(1) = 2\) and \(w(2) = 1\). Then the maximal chain \(C_w\) in \(\Pi_3\) corresponding to \(w\) is \((1 \mid 2 \mid 3) - (1 \mid 23) - (123)\). By identifying \(U_{45}, U_{123}, U_{67}\) with 1, 2, 3 in this order, one can get
\[\tilde{C}_w = U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}.\]

Since \([U_{45/23/67}, U_{45/123/67}]\) is isomorphic to a boolean algebra of the set of order 1, one can take
\[\tilde{C} = U_{45/23/67} \prec U_{45/123/67}.\]

Thus \(C_{D, w}\) is the chain
\[\hat{0} \prec U_{67} \prec U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}.\]

The upper interval \((U_{67}, \hat{1})\) is shown in Figure 3.5 and the chain \(\overline{C}_{D, w}\) is represented by thick lines.
The following lemma provides the relationship between the shelling-trapped decompositions of $[n]$ containing $F$ and the shelling-trapped decompositions of $F \cup \{v\}$.

**Lemma 3.4.11.** Let $\Delta$ be a shellable simplicial complex and let $F$ be the last facet in the shelling order of $\Delta$. Then there is a one-to-one correspondence between

- pairs $(D, w)$ of shelling-trapped decompositions $D$ of $[n]$ over $\Delta$ containing $F$ and $w \in S_{|D|−1}$, and
- pairs $(\tilde{D}, \tilde{w})$ of shelling-trapped decompositions $\tilde{D}$ of $F \cup \{v\}$ over $\Delta_F$ and $\tilde{w} \in S_{|\tilde{D}|−1}$.

Moreover, one can choose $C_{D,w}$ and $C_{\tilde{D},\tilde{w}}$ so that $C_{D,w} - U_F$ corresponds to $C_{\tilde{D},\tilde{w}}$ under the homotopy equivalence in Lemma 3.4.6.

**Proof.** Let $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ be a shelling-trapped decomposition of $[n]$ over $\Delta$ with $F_{i_p} = F$ and let $w$ be a permutation in $S_{p−1}$. Then one can see that $\tilde{F}_{i_j} = (F_{i_j} \cap F) \cup \{v\}$ are facets of $\Delta_F$ of type (B) for $j = 1, \ldots, p−1$ and $\tilde{F}_{i_1} < \cdots < \tilde{F}_{i_{p−1}}$. By Lemma 3.4.8, $\tilde{G}_{i_j} = G_{i_j} \cap F$ for all $j = 1, \ldots, p−1$. 

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There are two cases to consider:

Case 1. $\sigma_p \neq F$.

In this case, we will show

$$\tilde{D} = \{(\bar{\sigma}_p \cap F) \cup \{v\}, \tilde{F}_1, (\bar{\sigma}_1, \tilde{F}_1), \ldots, (\bar{\sigma}_{p-1}, \tilde{F}_{p-1})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$ over $\Delta_F$ ($\tilde{F}$ will be defined later).

Define

$$\tilde{\sigma}_j = \begin{cases} (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, p - 1, \\ \sigma_j & \text{for } j = p. \end{cases}$$

For $j = 1, \ldots, p - 1$, $\tilde{G}_{i_j} \subseteq \tilde{\sigma}_j \subseteq \tilde{F}_{i_j}$ since $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$.

Since $G_{i_p} \subseteq \sigma_p$, it must be that $\sigma_p$ is an intersection of some old walls of $F$. Thus one can find a family $\mathcal{G}$ of facets of $\Delta$ such that $\sigma_p = \bigcap_{F' \in \mathcal{G}} (F' \cap F)$ and $F' \cap F < F$ for all $F' \in \mathcal{G}$. Since $|F' \cup F| = |F'| + 1 < n$, one knows that $F' \cap F$ is a facet of $\Delta_F$ of type (A) for all $F' \in \mathcal{G}$. Let $\tilde{F} = F_k \cap F$ be the last facet in the family $\{F' \cap F | F' \in \mathcal{G}\}$ (pick $k$ as small as possible). Since all facets of $\Delta_F$ occurring earlier than $\tilde{F}$ have the form $F \cap F_i$ such that $F_i < F_k$ and $F_i \cap F \ll F$, one can see $\tilde{G} \subseteq \tilde{\sigma}_p \subseteq \tilde{F}$.

Thus $\tilde{D}$ is a shelling-trapped decomposition of $F \cup \{v\}$ over $\Delta_F$. Also one can define $\tilde{w} \in \mathfrak{S}_{p-1}$ by

$$\tilde{w}(j) = \begin{cases} w(j - 1) & \text{if } 1 < j \leq p - 1, \\ w(p - 1) & \text{if } j = 1. \end{cases}$$

Case 2. $\sigma_p = F$.

In this case, we claim that

$$\tilde{D} = \{(\bar{\sigma}_1, \tilde{F}_{i_1}), \ldots, (\bar{\sigma}_k \cup \{v\}, \tilde{F}_{i_k}), \ldots, (\bar{\sigma}_{p-1}, \tilde{F}_{i_{p-1}})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$.

Let $k = w(1)$. Define

$$\tilde{\sigma}_j = \begin{cases} \sigma_j \cap F & \text{for } j = k, \\ (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, k, \ldots, p - 1. \end{cases}$$
Then
\[ \tilde{G}_{ik} \subseteq \tilde{\sigma}_k = \sigma_k \cap F \subseteq \tilde{F}_{ik} \]
and
\[ \tilde{G}_{ij} \subseteq \tilde{\sigma}_j = (\sigma_j \cap F) \cup \{v\} \subseteq \tilde{F}_{ij} \]
for \( j = 1, \ldots, k, \ldots, p \). Thus \( \tilde{D} \) is a shelling-trapped decomposition of \( F \cup \{v\} \).

Also one can define \( \tilde{w} \in \mathfrak{S}_{p-2} \) as follows:
\[
\tilde{w}(j) = \begin{cases} 
  w(j + 1) & \text{if } w(j + 1) < k, \\
  w(j + 1) - 1 & \text{if } w(j + 1) > k.
\end{cases}
\]

Conversely, let
\[
\tilde{D} = \{([F \cup \{v\}] - \tilde{\sigma}_1, \tilde{F}_{i1}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, \tilde{F}_{ip})\}
\]
be a shelling-trapped decomposition of \( F \cup \{v\} \), where \( \tilde{F}_{ij} < \cdots < \tilde{F}_{ip} \) are facets of \( \Delta_F \), and let \( \tilde{w} \) be a permutation in \( \mathfrak{S}_{p-1} \). There is at most one \( \tilde{F}_{ij} \) that does not contain \( v \) because \( \tilde{F}_{ij} \cup \tilde{F}_{ik} = F \cup \{v\} \) for all \( j \neq k \). Since \( \tilde{F}_{i1} < \cdots < \tilde{F}_{ip} \) and the facets without \( v \) appear earlier than the ones with \( v \), there are two possible cases.

Case 1. \( v \notin \tilde{F}_{i1} \) and \( v \in \tilde{F}_{ij} \) for \( j = 2, \ldots, p \).

In this case, \( v \notin \tilde{\sigma}_p \), i.e., \( v \in [F \cup \{v\}] - \tilde{\sigma}_p \). One can show that a family
\[
D = \{([F \cup \{v\}] - \tilde{\sigma}_1, F_{i1}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, F_{ip}), ([n] - \tilde{\sigma}_1, F)\},
\]
where \( F_{ij} = (\tilde{F}_{ij} \cup F) - \{v\} \) for \( j = 2, \ldots, p \), is a shelling-trapped decomposition of \( [n] \) and \( \tilde{w} \in \mathfrak{S}_{p-1} \) is defined by
\[
\tilde{w}(j) = \begin{cases} 
  \tilde{w}(j + 1) & \text{if } 1 \leq j < p - 1, \\
  \tilde{w}(1) & \text{if } j = p - 1.
\end{cases}
\]

Case 2. \( v \in \tilde{F}_{ij} \) for \( j = 1, \ldots, p \).

In this case, there is a \( k \) such that \( v \in [F \cup \{v\}] - \tilde{\sigma}_k \). One can show that a family
\[
D = \{(F - \tilde{\sigma}_1, F_{i1}), \ldots, (F - \tilde{\sigma}_p, F_{ip}), ([n] - F, F)\},
\]

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where $F_{ij} = (\bar{F}_{ij} \cup F) - \{v\}$ for $j = 2, \ldots, p$, is a shelling-trapped decomposition of $[n]$ and $w \in S_p$ can be defined by
\[
w(j) = \begin{cases} 
\bar{w}(j - 1) & \text{if } 1 < j \text{ and } \bar{w}(j - 1) < k, \\
\bar{w}(j - 1) + 1 & \text{if } 1 < j \text{ and } \bar{w}(j - 1) \geq k, \\
k & \text{if } j = 1.
\end{cases}
\]

Proof of the second assertion is straightforward. \hfill \Box

**Example 3.4.12.** Let $\Delta$ be the shelling simplicial complex in Example 3.4.9. In Example 3.4.10, $C_{D,w}$ is the chain
\[
\hat{0} \prec U_{67} \prec U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}
\]
for the shelling-trapped decomposition
\[
D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}
\]
of $\{1, 2, 3, 4, 5, 6, 7\}$ and the permutation $w$ in $S_2$ with $w(1) = 2$ and $w(2) = 1$.

Since $67 = \bar{F}_7$, the corresponding shelling-trapped decomposition $\bar{D}$ of the set $\{1, 2, 3, 4, 5, v\}$ is
\[
\bar{D} = \{(45, \bar{F}_1 = 123v), (123v, \bar{F}_6 = 145v)\}
\]
and the corresponding permutation $\bar{w} \in S_1$ is the identity.

The map $\psi$ from the proof of Lemma 3.4.6 sends the chain
\[
U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367}
\]
to the chain
\[
U_{23} \prec U_{45/23} \prec U_{45/123} \prec U_{45/123v}
\]
and this chain satisfies the conditions for $C_{\bar{D},\bar{w}}$.

The intersection lattice for $\Delta_F$ is shown in Figure 3.6 and the chain $C_{\bar{D},\bar{w}}$ is represented by thick lines.

**Proof of Theorem 3.4.1.** By Lemma 3.4.5, it is enough to establish the case when $\bar{\sigma} = \cup_{i=1}^{\ell} \bar{F}_i = [n]$. Since every chain $C_{D,w}$ is saturated, it is enough to show
that \( \hat{\Delta}(L_\Delta) \), the simplicial complex obtained after removing the corresponding simplices for all pairs \((D,w)\), is contractible. We use induction on the number \( q \) of facets of \( \Delta \).

**Base case**: \( q = 2 \). If \( \Delta \) has only two facets \( F_1 \) and \( F_2 \) and \( \overline{F_1} \cup \overline{F_2} = [n] \), then \( F_2 \) has only one element and \( G_2 = \emptyset \). It is easy to see that the order complex \( \Delta(\overline{L_\Delta}) \) is homotopy equivalent to \( S^0 \) and \( D = \{([n], F_2)\} \) is the only shelling-trapped decomposition of \([n]\), while \( \overline{C}_{D,\emptyset} = (U_{\overline{\mathcal{F}_2}}) \) is the corresponding saturated chain. Therefore, \( \hat{\Delta}(L_\Delta) \) is contractible when \( q = 2 \).

**Inductive step**. Now, assume that \( \hat{\Delta}(L_\Delta) \) is contractible for all shellable simplicial complexes \( \Delta \) with less than \( q \) facets. For simplicity, denote \( L = L_\Delta \). Let \( F = F_q \) be the last facet in the shelling order of \( \Delta \) and \( H = U_\mathcal{F} \). Let \( L' \) be the intersection lattice for \( \Delta' \), where \( \Delta' \) is the simplicial complex generated by the facets \( F_1, \ldots, F_{q-1} \).

Let \( \overline{L}_{\geq H} \) denote the subposet of elements in \( \overline{L} \) which lie weakly above \( H \). Consider the decomposition of \( \hat{\Delta}(L) = X \cup Y \), where \( X \) is the simplicial complex obtained by removing all simplices corresponding to chains \( \overline{C}_{D,w} \) and \( \overline{C}_{D,w} - H \) from \( \Delta(\overline{L}_{\geq H}) \) for all \( \overline{C}_{D,w} \) containing \( H \), and \( Y \) is the simplicial complex obtained.
by removing all simplices corresponding to chains \( C_{D,w} \) not containing \( H \) from \( \Delta(\mathcal{L} - \{H\}) \). Our goal will be to show that \( X, Y \) and \( X \cap Y \) are all contractible, and hence so is \( X \cup Y (= \hat{\Delta}(L)) \).

**Step 1. Contractibility of \( X \)**

Since \( X \) has a cone point \( H \), it is contractible.

**Step 2. Contractibility of \( Y \)**

Define the closure relation \( \pi \) on \( \mathcal{L} \) which sends a subspace to the join of the elements covering 0 which lie below it except \( H \). Then the closed sets form a sublattice of \( \mathcal{L} \), which is the intersection lattice \( \mathcal{L}' \) for the diagonal arrangement corresponding to \( \Delta' \). It is known that the inclusion of closed sets \( \mathcal{L} \cap \mathcal{L}' \hookrightarrow \mathcal{L} - \{H\} \) is a homotopy equivalence (see \([8, \text{Lemma 7.6}] \)). We have to consider the following two cases:

**Case 1.** \( S := \bigcup_{i=1}^{q-1} F_i \neq [n] \),

Then \( \mathcal{L} \cap \mathcal{L}' = \mathcal{L} - \{0\} \) since \( U_S \in \mathcal{L} \). Since \( \mathcal{L} \cap \mathcal{L}' \) has a cone point \( U_S \), it is contractible. Since \( S \neq [n] \), there is no shelling-trapped decomposition of \([n]\) for \( \Delta' \). Thus \( Y \) is contractible.

**Case 2.** \( S := \bigcup_{i=1}^{q-1} F_i = [n] \),

In this case, \( \mathcal{L} \cap \mathcal{L}' = \mathcal{L}' \). Moreover, \( \hat{\Delta}(\mathcal{L}') \) is homotopy equivalent to \( Y \) since every element in a chain \( C_{D,w} \) in \( \hat{\Delta}(\mathcal{L} - \{H\}) \) is fixed under \( \pi \). Since \( \Delta' \) has \( q - 1 \) facets, the induction hypothesis implies that \( \hat{\Delta}(\mathcal{L}') \) is contractible and hence so is \( Y \).

**Step 3. Contractibility of \( X \cap Y \)**

Note that \( X \cap Y \) is obtained by removing simplices corresponding to \( C_{D,w} - H \) for all \( C_{D,w} \) containing \( H \) from \( \mathcal{L}_{>H} \). By Lemma 3.4.8, \( (\mathcal{L})_{>H} \) is isomorphic to the proper part of the intersection lattice \( \mathcal{L}_F \) for the diagonal arrangement corresponding to \( \Delta_F \) on \( F \cup \{v\} \). Also, Lemma 3.4.11 implies that \( X \cap Y \) is isomorphic to \( \hat{\Delta}(\mathcal{L}_F) \), where \( \hat{\Delta}(\mathcal{L}_F) \) is obtained by removing simplices corresponding to \( C_{\tilde{D},\tilde{w}} \) for all shelling-trapped decomposition \( \tilde{D} \) of \( F \cup \{v\} \) and \( \tilde{w} \in \mathcal{S}_{|\tilde{D}| - 1} \) from \( \mathcal{L}_F \). Since \( \Delta_F \) has fewer facets than \( \Delta \), the induction hypothesis implies \( \hat{\Delta}(\mathcal{L}_F) \) is
contractible and hence \( X \cap Y \) is also contractible.

**Example 3.4.13.** Let \( \Delta \) be a simplicial complex in Figure 2.1(c). Then \( F_1 = 123 \), \( F_2 = 234 \), \( F_3 = 35 \) and \( F_4 = 45 \) is a shelling of \( \Delta \) and

\[
G_1 = 123, \quad G_2 = 23, \quad G_3 = 3, \quad \text{and} \quad G_4 = \emptyset.
\]

Let \( \hat{\sigma} = 12345 \). Then \( \{(12345, F_4)\} \) and \( \{(45, F_1), (123, F_4)\} \) are two possible (unordered) shelling-trapped decompositions of \( \hat{\sigma} \) (see Table 3.2). Thus Theorem 3.4.1 implies \( \Delta(\hat{0}, U_{12345}) \) is homotopy equivalent to a wedge of two circles. The intersection lattice \( L_\Delta \) and the order complex for its proper part are shown in Figure 3.7. Note that the chains \( C_{D,w} \) and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines.

From Theorem 2.3.2 and our results in this section, one can deduce the following.

**Corollary 3.4.14.** Let \( \Delta \) be a shellable simplicial complex on \([n]\). The singularity link of \( A_\Delta \) has the homotopy type of a wedge of spheres, consisting of \( p! \) spheres of dimension

\[
n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2
\]
Table 3.2: Shelling-trapped decompositions for $\Delta$

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of some subset of $[n]$.

Proof sketch. This is a straightforward, but tedious, calculation. One needs to understand homotopy types of $\Delta(\hat{0}, H)$ for $H \in L_\Delta$ by Theorem 2.3.2. Lemmas 3.4.2 and 3.4.4 reduce this to the case of $\Delta(\hat{0}, U_\sigma)$, which is described fully by Theorem 3.4.1. The rest is some bookkeeping about shelling-trapped decompositions.

Example 3.4.15. Let $\Delta$ be a simplicial complex in Figure 2.1(c). In Example 3.4.13, we show

$$F_1 = 123, F_2 = 234, F_3 = 35, F_4 = 45$$

is a shelling of $\Delta$ and

$$G_1 = 123, G_2 = 23, G_3 = 3, G_4 = \emptyset.$$  

Table 3.2 shows shelling-trapped decompositions $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of subsets of $\{1, 2, 3, 4, 5\}$ with corresponding dimensions

$$n + p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| - 2.$$  

Therefore Corollary 3.4.14 shows that the singularity link of $A_\Delta$ is homotopy equivalent to a wedge of three 3-dimensional spheres and eight 2-dimensional spheres.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Shelling-trapped Decomp. & dim & Shelling-trapped Decomp. & dim \\
\hline
\{(45, F_1)\} & 3 & \{(15, F_2)\} & 3 \\
\{(145, F_2)\} & 3 & \{(124, F_3)\} & 2 \\
\{(1245, F_3)\} & 2 & \{(123, F_4)\} & 2 \\
\{(1234, F_4)\} & 2 & \{(1235, F_4)\} & 2 \\
\{(12345, F_4)\} & 2 & \{(45, F_1), (123, F_4)\} & 2 \\
\hline
\end{tabular}
\end{table}
3.5 Homology of diagonal arrangements

In this section, we show the homology version of Corollary 3.4.14 without using Theorem 3.1.1. This is what motivated us to prove the stronger Corollary 3.4.14 and eventually Theorem 3.1.1.

Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{F}$. Note that the field $\mathbb{F}$ need not have anything to do with the field $\mathbb{R}$ or $\mathbb{C}$ where the arrangement lives.

Let $\Gamma$ be a simplicial complex on a vertex set $[n] = \{1, 2, \ldots, n\}$. The Stanley-Reisner ideal $I_\Gamma$ is defined to be the ideal of $S$ generated by the set of monomials
\[
\{x_{i_1}x_{i_2}\cdots x_{i_r} : i_1 < i_2 < \cdots < i_r, \{i_1, i_2, \ldots, i_r\} \notin \Gamma\},
\]
and the Stanley-Reisner ring $\mathbb{F}[\Gamma]$ is defined to be $\mathbb{F}[\Gamma] := S/I_\Gamma$ (see [34]).

Note that $I_\Gamma$ is a monomial ideal, i.e., an ideal generated by monomials and $\mathbb{F}[\Gamma]$ is a vector space over $\mathbb{F}$ with basis
\[
\{x_{e_1}^{e_{i_1}}x_{e_2}^{e_{i_2}}\cdots x_{e_r}^{e_{i_r}} : \{i_1, i_2, \ldots, i_r\} \in \Gamma, e_1, e_2, \ldots, e_r > 0\}.
\]
In particular, $\mathbb{F}[\Gamma]$ is a graded ring.

The homology $\text{Tor}$ group $\text{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})$ can be computed from a free resolution of $\mathbb{F}$ over $S/I$ by considering $\mathbb{F}$ as a trivial $S/I$-module $\mathbb{F} = (S/I)/(x_1, \ldots, x_n)$. Since $S/I$ is $\mathbb{N}^n$-graded (for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the corresponding graded piece is the linear span of the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$), this resolution may also be chosen $\mathbb{N}^n$-graded. For a monomial $x^\alpha$, we denote by $\text{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})_\alpha$ or $\text{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})_{x^\alpha}$ the $\alpha$-graded piece of $\text{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})$.

Peeva, Reiner and Welker [30] show the following proposition.

**Proposition 3.5.1.** Let $\Delta$ be a simplicial complex on $[n] = \{1, 2, \ldots, n\}$. Then
\[
H_i(\mathcal{V}_{\Delta}^\alpha; \mathbb{F}) = \text{Tor}_i^{S/I}(\mathbb{F}, \mathbb{F})_{x_1 \cdots x_n},
\]

There are several ways to evaluate $\text{Tor}_i^{S/I}(\mathbb{F}, \mathbb{F})_\alpha$ in commutative algebra. We give one of them which uses Poincaré series.

The multigraded Poincaré series of $\mathbb{F}$ is
\[
\text{Poin}^\mathbb{F}_{S/I}(t, x) = \sum_{i \geq 0, \alpha \in \mathbb{N}^n} \dim_\mathbb{F} \text{Tor}_i^{S/I}(\mathbb{F}, \mathbb{F})_\alpha t^i x^\alpha.
\]

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It was proved by Serre that
\[
\operatorname{Poin}^F_{S/I}(t, x) \leq \frac{\prod_{i=1}^n(1 + tx_i)}{1 - tQ_{S/I}(t, x)},
\]
where the inequality means coefficient-wise comparison of power series and
\[
Q_{S/I}(t, x) = \sum_{i \geq 0, \alpha \in \mathbb{N}^n} \dim F \dim (S/I)_{\alpha} t^i x^\alpha.
\]

A ring \(S/I\) is called \textit{Golod} if equality holds in Serre’s inequality (3.1). It was shown by Golod that \(S/I\) is Golod exactly when certain homology operations (Massey operations) vanish in the Koszul complex computing \(\operatorname{Tor}^S_i(S/I, F)\) (see [21]).

It is known that if a homogeneous ideal \(I\) has a \textit{linear resolution} as an \(S\)-module, then \(S/I\) is Golod (see [2]). Also Herzog, Reiner and Welker [22] proved that, if \(I\) is a \textit{componentwise linear ideal}, then the ring \(S/I\) is Golod. They also observed that, for squarefree monomial ideals \(I = I_\Gamma\), componentwise linearity is equivalent to the dual complex \(\Gamma^*\) being \textit{sequentially Cohen-Macaulay over} \(F\), a notion introduced by Stanley [34]. It is known that if \(\Delta\) is shellable, then it is sequentially Cohen-Macaulay over all fields \(F\).

If \(\Delta\) is a shellable simplicial complex, then \(F[\Delta^*]\) is Golod and hence
\[
\operatorname{Poin}^F_{F[\Delta^*]}(t, x) = \frac{\prod_{i=1}^n(1 + tx_i)}{1 - tQ_F(S/I)}.
\]

Therefore
\[
\dim_F H_i(\mathcal{Y}_{\Delta}^\circ; F) = \left[t^{i-2}x_1 \cdots x_n\right] \frac{\prod_{i=1}^n(1 + tx_i)}{1 - tQ_F(S/I)},
\]
where the right hand side means the coefficient of \(t^{i-2}x_1 \cdots x_n\) in the power series expansion of \(\prod_{i=1}^n(1 + tx_i)\).

One can make this more explicit when \(\Delta\) is a shellable complex with the shelling order \(F_1, \ldots, F_q\).

\textbf{Lemma 3.5.2.} Let \(\Delta\) be a simplicial complex and \(F\) be a simplex such that \(H_i(\Delta) = 0\) for all \(i < \dim F\). Suppose \(F \cap \Delta\) is pure of codimension 1 in \(F\). Then
\[
\dim_F H_i(\text{link}_{\Delta \cup \{F\}}(\sigma); F) = \dim_F H_i(\text{link}_{\Delta}(\sigma); F).
\]
unless $\sigma \subset F$, $\text{link}_{\Delta}(\sigma) \cap (F - \sigma) = \partial(F - \sigma)$ and $i = \dim(F - \sigma)$. In that case,

$$
\dim_F H_i(\text{link}_{\Delta \cup \{F\}}(\sigma); \mathbb{F}) = \dim_F H_i(\text{link}_{\Delta}(\sigma); \mathbb{F}) + 1.
$$

**Proof.** Adding a new simplex $F$ can’t change $\text{link}_{\Delta \cup \{F\}}(\sigma)$ from $\text{link}_{\Delta}(\sigma)$ unless $\sigma \subset F$. Thus let’s assume that $\sigma \subset F$. For the simplicity, let $\hat{\Delta} = \text{link}_{\Delta}(\sigma)$ and $\hat{F} = F - \sigma$. Then $\text{link}_{\Delta \cup \{F\}}(\sigma) = \hat{\Delta} \cup \hat{F}$. Consider the Mayer-Vietoris sequence of $\{\hat{\Delta}, \hat{F}\}$:

$$
\cdots \to H_i(\hat{\Delta} \cap \hat{F}) \to H_i(\hat{\Delta}) \oplus H_i(\hat{F}) \to H_i(\hat{\Delta} \cup \hat{F}) \to H_{i-1}(\hat{\Delta} \cap \hat{F}) \to \cdots.
$$

Note that $\hat{F} \notin \hat{\Delta}$, otherwise $\hat{F} \cup \sigma = F \in \Delta$. So, $\hat{\Delta} \cap \hat{F} \subset \partial \hat{F}$. If $\hat{\Delta} \cap \hat{F} \subsetneq \partial \hat{F}$, then $\hat{\Delta} \cap \hat{F}$ is acyclic and $H_i(\hat{\Delta} \cap \hat{F}) = 0$ for all $i$. By the Mayer-Vietoris sequence, $H_i(\hat{\Delta} \cup \hat{F}) = H_i(\hat{\Delta}) \oplus H_i(\hat{F}) = H_i(\hat{\Delta})$. Now assume that $\hat{\Delta} \cap \hat{F} \subset \partial \hat{F} = S^a$ where $a = \dim \hat{F} - 1$. If $i < a$, $H_i(\hat{\Delta} \cap \hat{F}) = 0$. Thus $H_i(\hat{\Delta} \cup \hat{F}) = H_i(\hat{\Delta}) = H_i(\hat{\Delta})$ by the Mayer-Vietoris sequence. For $i = a$, consider the Mayer-Vietoris sequence

$$
\cdots \to H_{a+1}(\hat{\Delta} \cap \hat{F}) \to H_{a+1}(\hat{\Delta}) \oplus H_{a+1}(\hat{F}) \to H_{a+1}(\hat{\Delta} \cup \hat{F}) \to H_a(\hat{\Delta} \cap \hat{F}) \to H_a(\hat{\Delta}) \oplus H_a(\hat{F}) \to H_a(\hat{\Delta} \cup \hat{F}) \to \cdots.
$$

Since $\hat{\Delta} \cap \hat{F} = S^a$, $H_{a+1}(\hat{\Delta} \cap \hat{F}) = 0$ and $H_a(\hat{\Delta} \cap \hat{F}) = \mathbb{F}$. Since $\hat{F}$ is a simplex, $H_{a+1}(\hat{F}) = H_a(\hat{F}) = 0$. Since $a = \dim \hat{F} - 1 < \dim F$, one has $H_a(\hat{\Delta}) = 0$ by assumption. Therefore we get a short exact sequence

$$
0 \to H_{a+1}(\hat{\Delta}) \to H_{a+1}(\hat{\Delta} \cup \hat{F}) \to \mathbb{F} \to 0.
$$

Hence we have

$$
\dim_F H_{a+1}(\hat{\Delta} \cup \hat{F}; \mathbb{F}) = \dim_F H_{a+1}(\hat{\Delta}; \mathbb{F}) + 1.
$$

□

Eagon and Reiner [14] show

$$
Q_{\mathbb{F}[\Delta^*]}(t, x) = \sum_{i \geq 0, \sigma \in \Delta} \dim_F \tilde{H}_{i-2}(\text{link}_{\Delta}(\sigma); \mathbb{F}) t^i x^{\sigma}.
$$

(3.2)

This result together with Lemma 3.5.2 gives the following Theorem, which is the homology version of Corollary 3.4.14.
Theorem 3.5.3. Let \( \Delta \) be a shellable simplicial complex and \( F_1, \ldots, F_q \) be a shelling of \( \Delta \). Then \( \dim_{\mathbb{F}} H_i(\mathcal{V}_{\Delta_i}; \mathbb{F}) \) is the number of ordered shelling-trapped decompositions \( ((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})) \) of some subset of \([n]\) satisfying

\[
i = n + p(2 - n) + \sum_{j=1}^p |F_{i_j}| - 2.
\]

Proof. Let \( \Delta_i \) is a shellable complex with shelling \( F_1, \ldots, F_i \). By the rearrangement lemma \([8, \text{Lemma } 2.7]\), we may assume \( \dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_i \).

Since \( \text{link}_{\Delta_i - 1}(\sigma) \cap (F_i - \sigma) = \partial(F_i - \sigma) \) if and only if \( G_i \subset \sigma \) for \( i = 2, \ldots, q \), Lemma 3.5.2 and Equation (3.2) give

\[
Q_{\mathcal{V}[\Delta^*]}(t, x) = \sum_{i=1}^{q} \sum_{G_i \subset \sigma \subset F_i} t^{\dim_{\mathbb{F}}(\text{link}_{\Delta_i}(\sigma)) + 2} x^{\sigma}
\]

\[
= \sum_{i=1}^{q} \sum_{G_i \subset \sigma \subset F_i} t^{\dim \text{link}_{\Delta_i}(\sigma)} x^{\sigma}
\]

\[
= \sum_{\sigma \in \Delta} \sum_{j \in J_{\sigma}} t^{|F_j| - n + |\sigma| + 2} x^{\sigma},
\]

where \( J_{\sigma} = \{ j : G_j \subset \sigma \subset F_j \} \). Thus

\[
[x_1 \cdots x_n] \prod_{i=1}^{n} \left( 1 + tx_i \right) \frac{1}{1 - tQ_{\mathcal{V}[\Delta^*]}(t, x)}
\]

\[
= \sum_{S \subset [n]} t^{n - |S|} \left[ x^S \right] \left( \sum_{\rho \geq 0} \left( \sum_{\sigma \in \Delta} \sum_{j \in J_{\sigma}} t^{\dim \text{link}_{\Delta_i}(\sigma)} x^{\sigma} \right)^\rho \right)
\]

\[
= \sum_{S \subset [n]} t^{n - |S|} \sum_{((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))} t^{\sum_{j=1}^p (|F_{i_j}| - n + 2)}
\]

where the last sum is over all ordered shelling-trapped decompositions of \( S \). After the change of the order of sums and suitable degree shift, we get the desired property. \( \square \)

3.6 \( K(\pi, 1) \) examples from matroids

In this section, we give some examples of diagonal arrangements \( \mathcal{A} \) where the complement \( \mathcal{M}_{\mathcal{A}} \) is \( K(\pi, 1) \), coming from rank 3 matroids.
One should note that an arrangement having any subspace of real codimension 1 (hyperplane) will have $\mathcal{M}_A$ disconnected. So one may assume without loss of generality that all subspaces have real codimension at least 2. Furthermore, if any maximal subspace $U$ in $A$ has codimension at least 3, then it is not hard to see that $\mathcal{M}_A$ is not $K(\pi, 1)$. Hence we may assume without loss of generality that all maximal subspaces have real codimension 2.

A hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ is simplicial if every chamber in $M_{\mathcal{H}}$ is a simplicial cone. Davis, Januszkiewicz and Scott [12] show the following theorem.

**Theorem 3.6.1.** Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $A$ be any arrangement of codimension-2 subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_A$ is $K(\pi, 1)$.

**Remark 3.6.2.** In order to apply this to diagonal arrangements, we need to consider hyperplane arrangements $\mathcal{H}$ which are subarrangements of the braid arrangement $B_n$ and also simplicial. It turns out (and we omit the straightforward proof) that all such arrangements $\mathcal{H}$ are direct sums of smaller braid arrangements. So we only consider $\mathcal{H} = B_n$ itself here.

**Corollary 3.6.3.** Let $A$ be a subarrangement of the 3-equal arrangement of $\mathbb{R}^n$ so that

$$A = \{U_{ijk} \mid \{i, j, k\} \in T_A\},$$

for some collection $T_A$ of 3-element subsets of $[n]$. Then $A$ satisfies the hypothesis of Theorem 3.6.1 (and hence $\mathcal{M}_A$ is $K(\pi, 1)$) if and only if every permutation $w$ in $\mathcal{S}_n$ has at least one triple in $T_A$ consecutive.

**Proof.** It is easy to see that there is a bijection between chambers of $B_n$ and permutations $w = w_1 \cdots w_n$ in $\mathcal{S}_n$. Moreover, one can see that each chamber has the form $x_{w_1} > \cdots > x_{w_n}$ with bounding hyperplanes

$$x_{w_1} = x_{w_2}, x_{w_2} = x_{w_3}, \ldots, x_{w_{n-1}} = x_{w_n}$$

and intersects the 3-equal subspaces $x_{w_i} = x_{w_{i+1}} = x_{w_{i+2}}$ for $i = 1, 2, \ldots, n-2$. □

We seek shellable simplicial complexes $\Delta$ for which $A_\Delta$ satisfies this condition.
If $\Delta$ is the independent set complex $\mathcal{I}(M)$ for some matroid $M$, then facets of $\Delta$ are bases of $M$. Simplicial complexes of this kind are called matroid complexes, and they are known to be shellable [6]. For a rank 3 matroid $M$, consider
\[
\mathcal{A}_{\mathcal{I}(M^\perp)} = \{U_{ijk} \mid \{i,j,k\} = [n] - B \text{ for some } B \in \mathcal{B}(M^\perp)\} \\
= \{U_{ijk} \mid \{i,j,k\} \in \mathcal{B}(M)\},
\]
where $M^\perp$ is the dual matroid of $M$. Note that adding a loop to $M$ does not change the structure of the intersection lattice for $\mathcal{A}_{\mathcal{I}(M^\perp)}$. Thus, if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 3.6.3, then the diagonal arrangement corresponding to the matroid in which all loops have been deleted has $K(\pi, 1)$ complement, and hence $\mathcal{A}_{\mathcal{I}(M^\perp)}$ has $K(\pi, 1)$ complement.

**Definition 3.6.4.** Let $M$ be a rank 3 matroid on $[n]$. Say $M$ is DJS if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 3.6.3.

The following example shows that matroid complexes are not DJS in general. Thus we look for some subclasses of matroid complexes which are DJS, and hence whose corresponding diagonal arrangements have $K(\pi, 1)$ complements; for these, Theorem 3.5.3 gives us the group cohomology $H^\bullet(\pi, \mathbb{Z})$.

**Example 3.6.5.** Let $M$ be a matroid on $\{1, 2, 3, 4, 5, 6\}$ which has three distinct parallel classes $\{1, 6\}$, $\{2, 4\}$ and $\{3, 5\}$. Then $M$ is self-dual and $\mathcal{I}(M^\perp)$ is a simplicial complex on $[6]$ whose facets are $123, 134, 145, 125, 236, 256, 346$ and $456$. But $w = 124356$ is a permutation that does not satisfy the condition of Corollary 3.6.3.

Recall that a matroid is simple if it has no loops nor parallel elements. The following proposition shows that rank 3 simple matroids are DJS.

**Proposition 3.6.6.** Let $M$ be a matroid of rank 3 on the ground set $[n]$. If $M$ does not have parallel elements, then $M$ is DJS. In particular, rank 3 simple matroids are DJS.

**Proof.** Without loss of generality, we may assume that $M$ is simple. $M$ is not DJS if and only if there is a permutation $w \in \mathcal{S}_n$ such that every consecutive triple is
not in \( B(M) \). Since \( M \) is simple, the latter statement is true if and only if each consecutive triple in \( w \) form a circuit, i.e., all elements lie on a rank 2 flat. But this is impossible since \( M \) has rank 3.

The following two propositions give some subclasses of matroids with parallel elements which are DJS.

**Proposition 3.6.7.** Let \( M \) be a rank 3 matroid on the ground set \([n]\) with no circuits of size 3. Let \( P_1, \ldots, P_k \) be the distinct parallel classes which have more than one element, and let \( N \) be the set of all non-loop elements which are not parallel with anything else. Then, \( M \) is DJS if and only if

\[
\left\lfloor \frac{|P_1|}{2} \right\rfloor + \cdots + \left\lfloor \frac{|P_k|}{2} \right\rfloor - k < |N| - 2.
\]

*Proof.* We may assume that \( M \) does not have loops. Since \( M \) does not have loops nor circuits of size 3, \( M \) is not DJS if and only if one can construct a permutation \( w \in S_n \) such that for each consecutive triple in \( w \) there are at least two elements which are parallel. This means if \( w_i \in N \), then \( w_{i-2}, w_{i-1}, w_{i+1} \) and \( w_{i+2} \) (if they exist) must be in the same parallel class. Such a \( w \) can be constructed if and only if

\[
\left\lfloor \frac{|P_1|}{2} \right\rfloor + \cdots + \left\lfloor \frac{|P_k|}{2} \right\rfloor - k \geq |N| - 2.
\]


A simplicial complex \( \Delta \) on \([n]\) is shifted if, for any face \( \sigma \) of \( \Delta \), replacing any vertex \( i \in \sigma \) by a vertex \( j < i \) with \( j \notin \sigma \) gives another face in \( \Delta \). A matroid \( M \) is shifted if its independent set complex is shifted. Klivans [24] shows that a rank 3 shifted matroid on the ground set \([n]\) is indexed by some set \( \{a,b,c\} \) with \( 1 \leq a < b < c \leq n \) as follows:

\[
B(M) = \{(a',b',c') : 1 \leq a' < b' < c' \leq n, a' \leq a, b' \leq b, c' \leq c\}.
\]

It is not hard to check the following.

**Proposition 3.6.8.** Let \( M \) be the shifted rank 3 matroid on the ground set \([n]\) indexed by \( \{a,b,c\} \). Then, \( M \) is DJS if and only if \( \left\lfloor \frac{c-b}{2} \right\rfloor < a \).

We have not yet been able to characterize all rank 3 matroids which are DJS.
Chapter 4

Flag enumerations of matroid base polytopes

4.1 Main results

For a matroid $M$ on $[n]$, a matroid base polytope $Q(M)$ is the polytope in $\mathbb{R}^n$ whose vertices are the incidence vectors of the bases of $M$. The polytope $Q(M)$ is a face of a matroid polytope first studied by Edmonds [15], whose vertices are the incidence vectors of all independent sets in $M$. In this chapter, we study flags of faces in matroid base polytopes, and their cd-index.

It is known that a face $\sigma$ of a matroid base polytope is the matroid base polytope $Q(M_\sigma)$ for some matroid $M_\sigma$ on $[n]$ (see [18] and Section 4.2 below). We show that $M_\sigma$ can be described using equivalence classes of factor-connected flags of subsets of $[n]$. As a result, one can associate a poset for each face of $Q(M)$:

**Theorem 4.1.1.** Let $M$ be a matroid on a ground set $[n]$. For a face $\sigma$ of the matroid base polytope $Q(M)$, one can associate a poset $P_\sigma$ defined as follows:

(i) the elements of $P_\sigma$ are the connected components of $M_\sigma$, and

(ii) for distinct connected components $C_1$ and $C_2$ of $M_\sigma$, $C_1 < C_2$ if and only if $C_2 \subset S \subset [n]$ and $\sigma \subset H_S$ implies $C_1 \subset S$,

where $H_S$ is the hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = r(S)$. 

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We find the conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane:

**Theorem 4.1.2.** Let $M$ be a rank $r$ matroid on $[n]$ and $H$ be a hyperplane in $\mathbb{R}^n$ given by $\sum_{e \in S} x_e = k$. Then $H$ decomposes $Q(M)$ into two matroid base polytopes if and only if

(i) $r(S) \geq k$ and $r(S^c) \geq r - k$,

(ii) if $I_1$ and $I_2$ are $k$-element independent subsets of $S$ such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$.

The cd-index $\Psi(Q)$ of a polytope $Q$, a polynomial in the noncommutative variables $c$ and $d$, is a very compact encoding of the flag numbers of a polytope $Q$. Generalizing the formula of the cd-index of a prism and a pyramid of a polytope given by Ehrenborg and Readdy [16], we give the following theorem:

**Theorem 4.1.3.** If $Q$ is a polytope and $H$ is a hyperplane in $\mathbb{R}^n$, then the cd-index of $Q$ satisfies

$$
\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\hat{Q}) \cdot c - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot d \cdot \Psi(\hat{Q}/\hat{\sigma}),
$$

where the sum is over all proper faces $\sigma$ of $Q$ intersecting both open halfspaces obtained by $H$ nontrivially (notations will be defined in Section 4.4).

We apply these theorems to the cd-index of a matroid base polytope for rank 2 matroids.

In Section 4.2, we describe faces of matroid base polytopes using data of matroids. In Section 4.3, we define hyperplane splits of matroid base polytopes and give conditions for hyperplane splits. Section 4.4 contains the definition of cd-index and how cd-index is changed when a polytope is cut by a hyperplane. We apply these to the cd-index of matroid base polytopes for rank 2 matroids in Section 4.5.
4.2 Matroid base polytopes

This section contains the description of faces of matroid base polytopes. In particular, we associate a poset for each face of a matroid base polytope.

We start with a precise characterization of matroid base polytopes. Let $\mathcal{B}$ be a collection of $r$-element subsets of $[n]$. For each subset $B = \{b_1, \ldots, b_r\}$, let

$$e_B = e_{b_1} + \cdots + e_{b_r} \in \mathbb{R}^n,$$

where $e_i$ is the $i$th standard basis vector of $\mathbb{R}^n$. The collection $\mathcal{B}$ is represented by the convex hull of these points

$$Q(\mathcal{B}) = \text{conv}\{e_B : B \in \mathcal{B}\}.$$ 

This is a convex polytope of dimension $\leq n-1$ and is a subset of the $(n-1)$-simplex

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0, x_1 + \cdots + x_n = r\}.$$ 

Gelfand, Goresky, MacPherson, and Serganova [19] show the following characterization of matroid base polytopes.

**Theorem 4.2.1.** $\mathcal{B}$ is the collection of bases of a matroid if and only if every edge of the polytope $Q(\mathcal{B})$ is parallel to an edge of $\Delta_n$.

For a rank $r$ matroid $M$ on a ground set $[n]$ with a set of bases $\mathcal{B}(M)$, the polytope $Q(M) := Q(\mathcal{B}(M))$ is called the **matroid base polytope** of $M$.

By the definition, the vertices of $Q(M)$ represent the bases of $M$. Since every edge of $\Delta$ has the form $\text{conv}\{re_\alpha, re_\beta\}$ for some $\alpha \neq \beta$, every edge of $Q(M)$ is parallel to a difference $e_\alpha - e_\beta$ of two standard basis vectors. For two bases $B$ and $B'$ in $\mathcal{B}(M)$, $e_B$ and $e_{B'}$ are connected by an edge if and only if $e_B - e_{B'} = e_\alpha - e_\beta$. Since the latter condition is equivalent to $B - B' = \{\alpha\}$ and $B' - B = \{\beta\}$, the edges of $Q(M)$ represent the basis exchange axiom. The basis exchange axiom gives the following equivalence relation on the ground set $[n]$ of the matroid $M$: $\alpha$ and $\beta$ are **equivalent** if there exist bases $B$ and $B'$ in $\mathcal{B}(M)$ with $\alpha \in B$ and $B' = (B - \{\alpha\}) \cup \{\beta\}$. The equivalence classes are called the **connected components** of $M$. The matroid $M$ is called **connected** if it has only one connected component.
Feichtner and Sturmfels [18] express the dimension of the matroid base polytope $Q(M)$ in terms of the number of connected components of $M$.

**Proposition 4.2.2.** Let $M$ be a matroid on $[n]$. The dimension of the matroid base polytope $Q(M)$ equals $n - c(M)$, where $c(M)$ is the number of connected components of $M$.

Theorem 4.2.1 implies that every face of a matroid base polytope is also a matroid base polytope. For a face $\sigma$ of $Q(M)$, let $M_\sigma$ denote the matroid on $[n]$ whose matroid base polytope is $\sigma$. For $\omega \in \mathbb{R}^n$, let $M_\omega$ denote the matroid whose bases $B(M)$ is the collection of bases of $M$ having minimum $\omega$-weight. Then $Q(M_\omega)$ is the face of $Q(M)$ at which the linear form $\sum_{i=1}^{n} \omega_i x_i$ attains its minimum.

Let $\mathcal{F}(\omega)$ denote the unique flag of subsets

$$\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]$$

for which $\omega$ is constant on each set $S_i - S_{i-1}$ and $\omega|_{S_i - S_{i-1}} < \omega|_{S_{i+1} - S_i}$. The weight class of a flag $\mathcal{F}$ is the set of vectors $\omega$ such that $\mathcal{F}(\omega) = \mathcal{F}$. Ardila and Klivans [1] show that $M_\omega$ depends only on $\mathcal{F}(\omega)$, and hence one can call it $M_\mathcal{F}$. They also give the following description of $M_\mathcal{F}$.

**Proposition 4.2.3.** If $\mathcal{F}$ is a flag of subsets

$$\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n],$$

then

$$M_\mathcal{F} = \bigoplus_{i=1}^{k+1} (M|_{S_i})/S_{i-1}.$$ 

A flag $\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$ is called factor-connected if the matroids $(M|_{S_i})/S_{i-1}$ are connected for all $i = 1, \ldots, k+1$. Proposition 4.2.3 together with Proposition 4.2.2 shows that the dimension of $Q(M_\mathcal{F})$ is $n-k-1$ if $\mathcal{F}$ is factor-connected. In particular, facets of $Q(M)$ correspond to factor-connected flags of the form $\emptyset \subset S \subset [n]$. Feichtner and Sturmfels [18] show that there are two types of facets of $Q(M)$:
(i) a facet corresponding to a factor-connected flag $\emptyset \subset F \subset [n]$ for some flat $F$ of $M$ (in this case, the facet is called a flacet),

(ii) a facet corresponding to a factor-connected flag $\emptyset \subset S \subset [n]$ for an $(n-1)$-subset $S$ of $[n]$.

**Proposition 4.2.4.** Let $M$ be a connected matroid on $[n]$ and

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

be a factor-connected flag. Then the matroid $(M|_{S_{j+1}})/S_{j-1}$ has at most two connected components for $1 \leq j \leq k$.

(i) If it has one connected component, the flag

$$\mathcal{G} = \{\emptyset =: S_0 \subset \cdots \subset S_{j-1} \subset S_{j+1} \subset \cdots \subset S_{k+1} := [n]\}$$

is factor-connected and $Q(M_G)$ covers $Q(M_F)$ in the face lattice of $Q(M)$.

(ii) If it has two connected components, then they are $S_{j+1} - S_j$ and $S_j - S_{j-1}$.

Moreover, the flag

$$\mathcal{F}' = \{\emptyset =: S_0 \subset \cdots \subset S_{j-1} \subset S'_j \subset S_{j+1} \subset \cdots \subset S_{k+1} := [n]\},$$

where $S'_j = S_{j-1} \cup (S_{j+1} - S_j)$, is factor-connected and $Q(M_{\mathcal{F}'}) = Q(M_{\mathcal{F}})$.

**Proof.** For $j = 1, \ldots, k$, we have $(M|_{S_{j+1}})/S_j = [(M|_{S_{j+1}})/S_{j-1}]/(S_j - S_{j-1})$ and $(M|_{S_j})/S_{j-1} = [(M|_{S_{j+1}})/S_{j-1}]/S_j$. The first assertion follows from [29, Proposition 4.2.10], and the other assertions are obtained from [29, Proposition 4.2.13] and Proposition 4.2.3.

Two factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ are said to be **equivalent** if there is a sequence of factor-connected flags $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k = \mathcal{F}'$ such that $\mathcal{F}_i$ is obtained from $\mathcal{F}_{i-1}$ by applying Proposition 4.2.4(ii) for $i = 1, \ldots, k$. We write $\mathcal{F} \sim \mathcal{F}'$ when factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

**Lemma 4.2.5.** Let $M$ be a connected matroid on $[n]$ and $X, Y$ be disjoint subsets of $[n]$ such that $X \cup Y = [n]$. Then $\mathcal{B}((M|_X) \oplus (M/X))$ and $\mathcal{B}((M|_Y) \oplus (M/Y))$ are disjoint.
Proof. Since $X \cap Y = \emptyset$ and $X \cup Y = [n]$, one has $r(X) + r(Y) \geq r([n]) = r(M)$. If $r(X) + r(Y) = r(M)$, then $|B \cap X| = r(X)$ and $|B \cap Y| = r(Y)$ for all $B \in \mathcal{B}(M)$, and hence $M = M|_{X} \oplus M|_{Y}$. But then $M$ is not connected. Thus $r(X) + r(Y) > r(M)$. Therefore there is no base which has $r(X)$ elements in $X$ and $r(Y)$ elements in $Y$, i.e., $\mathcal{B}((M|_{X}) \oplus (M/X))$ and $\mathcal{B}((M|_{Y}) \oplus (M/Y))$ are disjoint. \qed

The following proposition shows that the equivalence classes of factor-connected flags characterize faces of a matroid base polytope.

**Proposition 4.2.6.** Let $M$ be a connected matroid on $[n]$. If $\mathcal{F}$ and $\mathcal{F}'$ are two factor-connected flags of subsets of $[n]$ given by

$$\mathcal{F} = \{\emptyset : S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\},$$

$$\mathcal{F}' = \{\emptyset : T_0 \subset T_1 \subset \cdots \subset T_l \subset T_{l+1} := [n]\},$$

then $M_{\mathcal{F}} = M_{\mathcal{F}'}$ if and only if $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

**Proof.** If $\mathcal{F} \sim \mathcal{F}'$, then $M_{\mathcal{F}} = M_{\mathcal{F}'}$ from Proposition 4.2.4.

For the other direction, suppose that $M_{\mathcal{F}} = M_{\mathcal{F}'}$. Then $\mathcal{F}$ and $\mathcal{F}'$ have the same length since $\dim Q(M_{\mathcal{F}}) = n - k - 1$ and $\dim Q(M_{\mathcal{F}'}) = n - l - 1$.

We will use induction on $k$. Without loss of generality, we may assume that $S_1 \neq T_1$. We claim that

$$T_1 = S_m - S_{m-1} \text{ for some } m > 1. \quad (4.1)$$

This claim is going to be used both in the base case and the inductive step. Let $m$ be the smallest index such that $T_1 \cap S_m \neq \emptyset$ and let $\alpha \in T_1 \cap S_m$. Suppose that $T_1$ is not contained in $S_m$, i.e., there is an element $\beta \in T_1 - S_m$. Since $M|_{T_1}$ is connected, there are $B'_1, \tilde{B}'_1 \in \mathcal{B}(M|_{T_1})$ such that $\alpha \in B'_1$ and $\tilde{B}'_1 = B'_1 - \{\alpha\} \cup \{\beta\}$. Choose $B'_j \in \mathcal{B}((M|_{T_j})/T_{j-1})$ for $j = 2, \ldots, k + 1$. Then

$$B' = B'_1 \cup B'_2 \cup \cdots \cup B'_{k+1} \in \mathcal{B}(M_{\mathcal{F}'}) ,$$

$$\tilde{B}' = \tilde{B}'_1 \cup B'_2 \cup \cdots \cup B'_{k+1} \in \mathcal{B}(M_{\mathcal{F}'}) .$$

But either $B'$ or $\tilde{B}'$ is not a base of $M_{\mathcal{F}}$ since

$$|\tilde{B}' \cap (S_m - S_{m-1})| = |B' \cap (S_m - S_{m-1})| - 1.$$
This contradicts the assumption \( M_\mathcal{F} = M_\mathcal{F}' \). Therefore \( T_1 \subset S_m \). Now suppose 
\((S_m - S_{m-1}) - T_1 = \emptyset \), i.e., there is an element \( \gamma \in (S_m - S_{m-1}) - T_1 \). Since 
\((M|_{S_m}) / S_{m-1}\) is connected, there are bases \( B_m, \tilde{B}_m \) of \((M|_{S_m}) / S_{m-1}\) satisfying 
\( \alpha \in B_m \) and \( \tilde{B}_m = B_m - \{\alpha\} \cup \{\gamma\} \). If \( B_j \in \mathcal{B}((M|_{S_j}) / S_{j-1}) \) for all \( j \neq m \), then 
\[
B = B_1 \cup \cdots \cup B_{k+1} \in \mathcal{B}(M_\mathcal{F}),
\]
\[
\tilde{B} = B_1 \cup \cdots \cup B_{m-1} \cup \tilde{B}_m \cup B_{m+1} \cup \cdots \cup B_{k+1} \in \mathcal{B}(M_\mathcal{F}).
\]

But at least one of them is not in \( \mathcal{B}(M_\mathcal{F}) \) since \( |\tilde{B} \cap T_1| = |B \cap T_1| - 1 \). This is 
impossible because \( M_\mathcal{F} = M_\mathcal{F}' \). Thus \((S_m - S_{m-1}) - T_1 = \emptyset \), i.e., \( T_1 = S_m - S_{m-1} \).

**Base case:** \( k = 1 \). Equation (4.1) gives \( T_1 = S_2 - S_1 \) and \( T_1 \cup S_1 = [n] \). Since 
\( M \) has two connected components by Lemma 4.2.5, \( \mathcal{F} \) and \( \mathcal{F}' \) are equivalent by 
Proposition 4.2.4(ii).

**Inductive step.** Now suppose that \( k > 1 \). Let \( \tilde{\mathcal{F}} \) be a flag 
\[
\emptyset =: S_0 \subset S_1 \subset S_1 \cup T_1 \subset S_2 \cup T_1 \subset \cdots \subset S_{m-1} \cup T_1 \subset S_{m+1} \cdots \subset S_{k+1} := [n].
\]

We claim that \( M|_{S_j \cup T_1} = M|_{S_j} \oplus M|_{T_1} \) for \( j = 1, \ldots, m - 1 \). Let \( B_j \in \mathcal{B}(M|_{S_j}) \) 
and \( B'_j \in \mathcal{B}(M|_{T_1}) \). Since \( M_\mathcal{F} = \oplus_{i=1}^{k+1}(M|_{S_i}) / S_{i-1} = \oplus_{i=1}^{k+1}(M|_{T_i}) / T_{i-1} = M_\mathcal{F}' \) and 
\( M|_{T_1} = (M|_{S_m}) / S_{m-1} \), there is a base \( B \in \mathcal{B}(M_\mathcal{F}) \) such that \( B \cap S_j = B_j \) and 
\( B \cap T_1 = B'_1 \). Thus \( B_j \cup B'_1 \in \mathcal{B}(M|_{S_j \cup T_1}) \), and hence 
\[
r(M|_{S_j \cup T_1}) = r(M|_{S_j}) + r(M|_{T_1}).
\]

By [27, Lemma 3], we have \( M|_{S_j \cup T_1} = M|_{S_j} \oplus M|_{T_1} \). Therefore 
\[
(M|_{S_j \cup T_1}) / S_1 = [M|_{T_1} \oplus M|_{S_j}] / S_1
\]
\[
= (M|_{S_m}) / S_{m-1}
\]
and 
\[
(M|_{S_i \cup T_1}) / (S_{i-1} \cup T_1) = [M|_{T_1} \oplus M|_{S_i}] / (S_{i-1} \cup T_1)
\]
\[
= (M|_{S_i}) / S_{i-1}
\]
for \( i = 2, \ldots, m - 1 \). Thus \( \tilde{\mathcal{F}} \) is factor-connected and \( M_\tilde{\mathcal{F}} = M_\mathcal{F} \), i.e., \( \tilde{\mathcal{F}} \sim \mathcal{F} \).

Also, one can show \( S_1 = T_{i} - T_{i-1} \) for some \( l > 1 \) and the chain \( \tilde{\mathcal{F}'} \) given by 
\[
\emptyset =: T_0 \subset T_1 \subset S_1 \cup T_1 \subset S_1 \cup T_2 \subset \cdots \subset S_1 \cup T_{i-1} \subset T_{i+1} \cdots \subset T_{k+1} := [n]
\]
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is factor-connected and $M_{\tilde{F}'} = M_{\tilde{F}}$, i.e., $\tilde{F}' \sim F'$.

By the induction assumption, we have $\tilde{F} \sim \tilde{F}'$ and hence $F$ and $F'$ are equivalent.

If $M$ is a matroid on $[n]$ and $S$ is a subset of $[n]$, then the hyperplane $H_S$ defined by $\sum_{e \in S} x_e = r(S)$ is a supporting hyperplane of $Q(M)$ and $Q(M) \cap H_S$ is the matroid base polytope for $(M|_S) \oplus (M/S)$. The next lemma tells us when a face of $Q(M)$ is contained in $H_S$.

**Lemma 4.2.7.** Let $M$ be a matroid on $[n]$ and $S$ be a subset of $[n]$. A face $\sigma$ of $Q(M)$ is contained in $H_S$ if and only if there is a factor-connected flag

$$F = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

such that $S \in F$ and $\sigma = Q(M_F)$.

**Proof.** If there is a flag $F$ containing $S$ and $\sigma = Q(M_F)$, then every base of $M_F$ is contained in $B((M|_S) \oplus (M/S))$ and hence $\sigma$ is contained in $H_S$.

For the converse, suppose $\sigma$ is contained in $H_S$. Since $\sigma$ is a face of $Q(M)$, there is a factor-connected flag

$$F' = \{\emptyset =: T_0 \subset T_1 \subset \cdots \subset T_k \subset T_{k+1} := [n]\},$$

such that $\sigma = Q(M_{F'})$ and $k = n - \dim \sigma - 1$. We claim that $T_i - T_{i-1}$ is contained in either $S$ or $[n] - S$ for $i = 1, \ldots, k + 1$. If $(M|_{T_i})/T_{i-1}$ contains a loop, then $T_i - T_{i-1}$ has only one element because $(M|_{T_i})/T_{i-1}$ is connected, hence is contained in either $S$ or $[n] - S$. Now assume $(M|_{T_i})/T_{i-1}$ does not contain a loop. Suppose there are $x \in (T_i - T_{i-1}) \cap S$ and $y \in (T_i - T_{i-1}) - S$. Since $(M|_{T_i})/T_{i-1}$ is connected, there exist bases $B_1$ and $B_2$ of $M_{F'}$ with $x \in B_1$ and $B_2 = (B_1 - \{x\}) \cup \{y\}$. Then $|B_1 \cap S| > |B_2 \cap S|$ and so $\sigma$ is not contained in $H_S$, which is a contradiction.

Construct a flag $F$ as follows: $S_j = S_{j-1} \cup (T_k - T_{k-1})$, where $k$ is the smallest index such that

(i) $T_k - T_{k-1} \subset S - S_j$ if $S - S_j \neq \emptyset$,  


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(ii) $T_k - T_{k-1} \subset [n] - S$ if $S - S_j = \emptyset$.

Then $M_F = M_{F'}$ and hence $\sigma = Q(M_F)$.

For a face $\sigma$ of $Q(M)$, let $L_\sigma$ be the poset of all subsets of $[n]$ which are contained in some factor-connected flag $F$ with $\sigma = Q(M_F)$ ordered by inclusion. The following lemma shows that $L_\sigma$ is a lattice.

**Lemma 4.2.8.** Let $M$ be a matroid on $[n]$ and $S, T \subset [n]$. If $Q(M)$ is contained in $H_S$ and $H_T$, then it is also contained in $H_{S \cap T}$.

**Proof.** Since $Q(M)$ is contained in $H_S$ and $H_T$, we have

$$r(S \cup T) + r(S \cap T) = r(S) + r(T)$$

(see [29, Lemma 1.3.1]). Suppose that $Q(M)$ is not contained in $H_{S \cap T}$, i.e., there is a base $B$ of $M$ such that $|B \cap (S \cap T)| < r(S \cap T)$. Then

$$|B \cap (S \cup T)| = |B \cap S| + |B \cap T| - |B \cap (S \cap T)|$$

$$> r(S) + r(T) - r(S \cap T) = r(S \cup T)$$

which is impossible. Therefore $Q(M)$ is contained in $H_{S \cap T}$.

Since $L_\sigma$ is a sublattice of the Boolean lattice $B_n$, it is distributive. The fundamental theorem for finite distributive lattices [35] shows that there is a finite poset $P$ for which $L_\sigma$ is the lattice of order ideals of $P$.

**Definition 4.2.9.** For a face $\sigma$ of $Q(M)$, define a poset $P_\sigma$ as follows:

(i) The elements of $P_\sigma$ are the connected components of $M_\sigma$, and

(ii) for distinct connected components $C_1$ and $C_2$ of $M_\sigma$, $C_1 < C_2$ if and only if

$$C_2 \subset S \subset [n] \text{ and } \sigma \subset H_S \text{ implies } C_1 \subset S.$$ 

Note that $P_\sigma$ is a well-defined poset. Reflexivity and transitivity are clear. Suppose $C_1$ and $C_2$ are distinct components with $C_1 < C_2$. Consider a minimal subset $S$ such that $C_2 \subset S$ and $\sigma \subset H_S$. Then $\sigma \subset H_{S - C_2}$ by Lemma 4.2.7. Since $C_1 \subset S - C_2$ and $C_2 \not\subset S - C_2$, we have $C_2 \not\subset C_1$. 

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Figure 4.1: The proper part of the face poset of $Q(M_{2,1,1})$

**Example 4.2.10.** Let $M_{2,1,1}$ be the rank 2 matroid on $[4] = \{1, 2, 3, 4\}$ whose unique non-base is 12 and let $\sigma$ be an edge of $Q(M_{2,1,1})$ connecting $e_{14}$ and $e_{24}$. Then the connected components of $M_\sigma$ are 12, 3 and 4. Since $\{1, 2, 3, 4\}$ is the only subset $S$ containing $\{3\}$ such that $\sigma \subset H_S$, $12 < 3$ and $4 < 3$ in $P_\sigma$. One can see that there are no other relations in $P_\sigma$. Figure 4.1 is the proper part of the face poset of $Q(M_{2,1,1})$ whose faces are labeled by corresponding posets and $P_\sigma$ is shown in the shaded box.

**Theorem 4.2.11.** Let $M$ be a matroid on $[n]$ and $\sigma$ be a face of $Q(M)$. Then $L_\sigma$ is the lattice of order ideals of $P_\sigma$.

**Proof.** Let $S$ be a set in $L_\sigma$. By Lemma 4.2.7, there is a factor-connected flag

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

such that $M_\sigma = M_\mathcal{F}$ and $S_i = S$ for some $i$. Then $S = \bigcup_{j=1}^{i} (S_j - S_{j-1})$. Suppose $S_i - S_{i-1} < S_j - S_{j-1}$ in $L_\sigma$ for some $j \leq i$. Since $\sigma \subset H_S$, $S_i - S_{i-1} \subset S$ and hence $S$ is an order ideal of $P_\sigma$. 

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Conversely, suppose $T$ is an order ideal of $P_\sigma$. Let $U$ be the intersection of all subsets $\tilde{T}$ such that $\sigma \subset H_\tilde{T}$ and $T \subset \tilde{T}$. By Lemma 4.2.8, $U$ lies in $L_\sigma$. Suppose $T \neq U$. By Lemma 4.2.7, there is a factor-connected flag

$$\mathcal{F} = \{\emptyset = U_0 \subset U_1 \subset \cdots \subset U_k \subset U_{k+1} := [n]\}$$

such that $\sigma = Q(M_\mathcal{F})$ and $U = U_j$ for some $j$. Since $T \neq U$ and $T$ is an order ideal, we may choose $\mathcal{F}$ so that $U_j - U_{j-1} \notin T$. Then $U_{j-1} \in L_\sigma$ and $T \subset U_{j-1}$ which is a contradiction. \hfill \Box

Billera, Jia and Reiner [4] define posets related to bases of matroids: For each $e \in B \in \mathcal{B}(M)$ the \textit{basic bond for $e$ in $B$} is the set of $e' \in [n] - B$ for which $B - \{e\} \cup \{e'\}$ is another base of $M$. Dually, for each $e' \in [n] - B$ the \textit{basic circuit for $e$ in $B$} is the set of $e \in B$ for which $B - \{e\} \cup \{e'\}$ is another base of $M$. Note that $e'$ lies in the basic bond for $e$ if and only if $e$ lies in the basic circuit of $e'$. Thus one can define the poset $P_B$ to be the poset whose Hasse diagram is the bipartite graph with vertex set $[n]$ and edges directed upward from $e$ to $e'$ whenever $e'$ is in the basic bond for $e$.

The next proposition shows that our poset $P_\sigma$ is the same as the poset $P_B$ of Billera, Jia and Reiner if $\sigma$ is a vertex $e_B$.

**Proposition 4.2.12.** If $B$ is a base of a matroid $M$ on $[n]$ and $\sigma$ is a vertex $e_B$ of $Q(M)$, then $P_\sigma = P_B$.

**Proof.** Since all connected components of $M_\sigma$ are singletons, $P_\sigma$ and $P_B$ have the same elements. If $e_1, e_2 \in B$, then they are not comparable in $P_\sigma$ since $e_B \in H_{e_1}$ and $e_B \notin H_{e_2}$. Also, $e_1, e_2 \notin B$ are not comparable in $P_\sigma$ because $e_B \in H_{B \cup \{e_1\}}$ and $e_B \notin H_{B \cup \{e_2\}}$. Now assume $e_1 \in B$ and $e_2 \notin B$. Then $e_2 \notin e_1$ since $e_B \in H_{e_1}$. We claim that $e_1 \prec_{P_\sigma} e_2$ if and only if $B' := B - \{e_1\} \cup \{e_2\} \in \mathcal{B}(M)$, i.e., $e_1 \prec_{P_B} e_2$. Assume $e_1 \prec_{P_B} e_2$, i.e., $e_2 \in S$ and $\sigma_B \in H_S$ implies $e_1 \in S$. Suppose $B'$ is not a base of $M$. Then $r(B') = r(M) - 1$. Thus we have $e_2 \in B', \sigma_B \in H_{B'}$, but $e_1 \notin B'$, which is a contradiction. Therefore $B'$ is a base of $M$. Conversely, assume $B' \in \mathcal{B}(M)$. Let $S$ be a subset of $[n]$ such that $e_2 \in S$ and $\sigma_B \in H_S$, i.e., $|B \cap S| = r(S)$. If $e_1 \notin S$, then $|B' \cap S| = |B \cap S| + 1 = r(S) + 1$, which is impossible. Thus $e_1 \prec_{P_\sigma} e_2$. \hfill \Box
4.3 Matroid base polytope decompositions

In this section, we define the hyperplane split of a matroid base polytope and give conditions when it occurs.

For a matroid $M$ on $[n]$, a matroid base polytope decomposition of $Q(M)$ is a decomposition $Q(M) = \bigcup_{i=1}^{t} Q(M_i)$ where

(i) each $M_i$ is a matroid on $[n]$, and

(ii) for each $i \neq j$, the intersection $Q(M_i) \cap Q(M_j)$ is a proper face of both $Q(M_i)$ and $Q(M_j)$.

A matroid base polytope decomposition of $Q(M)$ is called a hyperplane split of $Q(M)$ if $t = 2$. We also say that $Q(M)$ is decomposable if there is a matroid base polytope decomposition with $t \geq 2$, and indecomposable otherwise.

Consider a hyperplane split $Q(M_1) \cup Q(M_2)$ of a matroid base polytope $Q(M)$. Let $\sum_{i=1}^{n} a_i x_i = b$ be an equation defining the corresponding hyperplane $H$. Since $Q(M_1) \cap Q(M_2)$ is a matroid base polytope on $H$ and its edges are parallel to $e_i - e_j$ for some $i \neq j$, the only constraints on the normal vector $(a_1, a_2, \ldots, a_n)$ of $H$ are of the form $a_i = a_j$. Using the fact that $Q(M)$ is a subset of a simplex defined by $\sum_{i=1}^{n} x_i = r(M)$ and scaling the right hand side $b$, one can assume that $H$ is defined by $\sum_{e \in S} x_e = k$ for some subset $S$ of $[n]$.

The following result characterizes when hyperplane splits occur.

**Theorem 4.3.1.** Let $M$ be a rank $r$ matroid on $[n]$ and $H$ be a hyperplane defined by $\sum_{e \in S} x_e = k$. Then $H$ gives a hyperplane split of $Q(M)$ if and only if the following are satisfied:

(i) $r(S) > k$ and $r(S^c) > r - k$,

(ii) if $I_1$ and $I_2$ are $k$-element independent subsets of $S$ such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$.

**Remark 4.3.2.** Note that if $I$ is a $k$-element independent subset of $S$ and $J$ is an $(r - k)$-element independent subset of $S^c$, then $I$ is a base for $(M/J)|_{S^c}$ if and only if $J$ is a base for $(M/I)|_{S^c}$. Therefore one can see that the condition (ii) can be replaced with the following condition for $S^c$: 61
(ii') if $J_1$ and $J_2$ are $(r-k)$-element independent subsets of $S^c$ such that $(M/J_1)|_S$ and $(M/J_2)|_S$ have rank $k$, then $(M/J_1)|_S = (M/J_2)|_S$.

Proof of Theorem 4.3.1. Define $\mathcal{B}_k = \{ B \in \mathcal{B}(M) : |B \cap S| = k \}$. We will show that the condition (ii) holds if and only if $\mathcal{B}_k$ is a collection of bases of some matroid. Then the assertion follows from Theorem 4.2.1.

Suppose that the condition (ii) is true. Choose any bases $B_1$ and $B_2$ in $\mathcal{B}_k$ and $x \in B_1 - B_2$ (without loss of generality, we may assume $x \in B_1 \cap S$). Let $I_i = B_i \cap S$ and $J_i = B_i - S$ for $i = 1, 2$. Then the condition (ii) implies that there is a base $B = I_2 \cup J_1$ in $\mathcal{B}_k$. Since $B_1, B_2 \in \mathcal{B}$, there is $y \in B - B_1 \subset I_1 \subset B_2$ such that $B_3 = B - \{x\} \cup \{y\} \in \mathcal{B}$. Since $y \in I_2 \subset S$, $B_3 \in \mathcal{B}_k$. Thus $\mathcal{B}_k$ forms a collection of bases of a matroid.

Conversely suppose that $\mathcal{B}_k$ is a collection of bases of some matroid. Let $I_1$ and $I_2$ be $k$-element independent subsets of $S$ such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$. Choose $J_1 \in \mathcal{B}((M/I_1)|_{S^c})$ and $J_2 \in \mathcal{B}((M/I_2)|_{S^c})$. Then $B_1 = I_1 \cup J_1$ and $B_2 = I_2 \cup J_2$ are bases for $B$. We claim that $I_2 \cup J_1$ is also a base of $M$: this implies $\mathcal{B}((M/I_1)|_{S^c}) \subset \mathcal{B}((M/I_2)|_{S^c})$ and (ii) follows by symmetry. We use induction on the size of $I_1 - I_2$.

Base Case: If $|I_1 - I_2| = 0$, we have $I_2 \cup J_1 = B_1 \in \mathcal{B}$.

Inductive Step: Suppose $|I_1 - I_2| = l$ for some $l \leq k$. Choose an element $x \in I_1 - I_2 \subset B_1 - B_2$. Since $\mathcal{B}_k$ forms a matroid, there exist $y \in I_2 - I_1$ such that $B_3 = B_1 - \{x\} \cup \{y\} \in \mathcal{B}_k \subset \mathcal{B}$. Since $B_3 = (I_1 - \{x\} \cup \{y\}) \cup J_1$, we have $|(B_3 \cap S) - I_2| = l - 1$ and the induction hypothesis implies $I_2 \cup J_1 \in \mathcal{B}$.

When $M$ has rank 2, Theorem 4.3.1 can be rephrased in the following way.

Corollary 4.3.3. Let $M$ be a rank 2 matroid on $[n]$ and $H$ be a hyperplane defined by $\sum_{e \in S} x_e = 1$. Then $H$ gives a hyperplane split of $Q(M)$ if and only if $S$ and $S^c$ are both unions of at least two rank one flats.

4.4 The cd-index

In this section, we define the cd-index for Eulerian posets and give the relationship among cd-indices of polytopes when a polytope is cut by a hyperplane.
Let $P$ be a graded poset of rank $n + 1$ and $S$ be a subset of $[n]$. Define $f_P(S)$ to be the number of chains of $P$ whose ranks are exactly given by the set $S$. The function $f_P : 2^{[n]} \to \mathbb{N}$ is called the flag $f$-vector of $P$. The flag $h$-vector is defined by the identity

$$h_P(S) = \sum_{T \subset S} (-1)^{|S-T|} \cdot f_P(T).$$

Since this identity is equivalent to the relation

$$f_P(S) = \sum_{T \subset S} h_P(T),$$

the flag $f$-vector and the flag $h$-vector contain the same information.

For a subset $S$ of $[n]$ define the $\mathbf{ab}$-monomial $u_S = u_1 \ldots u_n$, where

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

The $\mathbf{ab}$-index of the poset $P$ is defined to be the sum

$$\Psi(P) = \sum_{S \subset [n]} h_P(S) \cdot u_S.$$

An alternative way of defining the $\mathbf{ab}$-index is as follows. For a chain $c := \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$, we give a weight $w_P(c) = w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} b & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ a - b & \text{otherwise}. \end{cases}$$

Define the $\mathbf{ab}$-index of the poset $P$ to be the sum

$$\Psi(P) = \sum_c w(c),$$

where the sum is over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in $P$. Recall that a poset $P$ is Eulerian if its Möbius function $\mu$ is given by $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ (see [35] for more details). One important class of Eulerian posets is face lattices of convex polytopes (see [26, 33]). It is known that the $\mathbf{ab}$-index of an Eulerian
poset \( P \) can be written uniquely as a polynomial of \( c = a + b \) and \( d = ab + ba \) (see [3]). When the \( ab \)-index can be written as a polynomial in \( c \) and \( d \), we call \( \Psi(P) \) the \textbf{cd-index} of \( P \). We will use the notation \( \Psi(Q) \) for the \textbf{cd-index} of the face poset of a convex polytope \( Q \).

Ehrenborg and Readdy [16] give formulas for the \textbf{cd-index} of a pyramid, a prism and a bipyramid of a polytope.

**Proposition 4.4.1.** Let \( Q \) be a polytope. Then

\[
\Psi(\text{Pyr}(Q)) = \frac{1}{2} \left[ \Psi(Q) \cdot c + c \cdot \Psi(Q) + \sum_{\sigma} \Psi(\sigma) \cdot d \cdot \Psi(Q/\sigma) \right],
\]
\[
\Psi(\text{Prism}(Q)) = \Psi(Q) \cdot c + \sum_{\sigma} \Psi(\sigma) \cdot d \cdot \Psi(Q/\sigma),
\]
\[
\Psi(\text{Bipyrr}(Q)) = c \cdot \Psi(Q) + \sum_{\sigma} \Psi(\sigma) \cdot d \cdot \Psi(Q/\sigma),
\]

where the sum is over all proper faces \( \sigma \) of \( Q \).

Note that the \textbf{cd-index} of Bipyrr\( (Q) \) is obtained from the fact that Bipyrr\( (Q) \) is the dual of the prism over the dual of \( Q \) and the \textbf{cd-index} of the dual polytope is obtained by writing every \textbf{ab}-monomial in reverse order.

Let \( Q \) be a polytope in \( \mathbb{R}^n \). Let \( H \) be a hyperplane in \( \mathbb{R}^n \) defined by \( l(x) = c \) and \( H^+ \) (resp. \( H^- \)) be the closed halfspace \( l(x) \geq c \) (resp. \( l(x) \leq c \)). Also recall that relint \( Q \) is the relative interior of a polytope \( Q \). For simplicity, let \( Q^+ := Q \cap H^+ \), \( Q^- := Q \cap H^- \), and \( \hat{Q} := Q \cap H \). The following theorem provides the relationship among \textbf{cd-indices} of polytopes \( Q, Q^+, Q^- \) and faces of \( \hat{Q} \).

**Theorem 4.4.2.** Let \( Q \) be a polytope in \( \mathbb{R}^n \) and \( H \) be a hyperplane in \( \mathbb{R}^n \). Then the following identity holds:

\[
\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\hat{Q}) \cdot c - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot d \cdot \Psi(\hat{Q}/\hat{\sigma}),
\]

where the sum is over all proper faces \( \sigma \) of \( Q \) intersecting both open halfspaces \( H^+ - H \) and \( H^- - H \) nontrivially.

**Proof.** Let

\[
c = \{\emptyset = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = Q\}
\]
be a chain in the face poset of $Q$. If there is a face $\sigma_i$ such that $\text{relint } \sigma_i \subset H^+ - H$ (resp. $\text{relint } \sigma_i \subset H^- - H$), then

$$c^+ = \{\emptyset < \sigma_1 \cap H^+ < \cdots < \sigma_k \cap H^+ < Q^+\}$$

(resp. $c^- = \{\emptyset < \sigma_1 \cap H^- < \cdots < \sigma_k \cap H^- < Q^-\}$)

is a corresponding chain in the face poset of $Q^+$ (resp. $Q^-$) and $w(c) = w(c^+)$ (resp. $w(c) = w(c^-)$).

Now suppose that $\text{relint } \sigma_i$ is contained in neither $H^+ - H$ nor $H^- - H$ for all $i = 1, \ldots, k$. Let $s$ be the smallest index such that $\sigma_i \not\subset H$ and $\hat{c}$ be the chain in the face poset of $\hat{Q}$ defined by

$$\hat{c} := c \cap H = \{\emptyset = \sigma_0 < \sigma_1 < \cdots < \sigma_{k+1} = \hat{Q}\}.$$ 

(i) Consider a chain $c^+$ in the face poset of $Q^+$ such that $c^+ \cap H$ and $\hat{c}$ are the same. Then $c^+$ is one of the following chains for some $j$ with $s \leq j \leq k+1$:

$$c_1^+ = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \sigma_j^+ < \cdots < \sigma_k^+ < \sigma_{k+1}^+ = Q^+\},$$

$$c_2^+ = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \hat{\sigma}_j < \sigma_j^+ < \cdots < \sigma_k^+ < \sigma_{k+1}^+ = Q^+\}.$$

Let $\sigma := \sigma_j$. Then $\text{relint } \sigma \cap (H^+ - H) \neq \emptyset$ and $\text{relint } \sigma \cap (H^- - H) \neq \emptyset$. Let $\hat{c}_1 := \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \hat{\sigma}\}$ and $\hat{c}_2 := \{\hat{\sigma} < \hat{\sigma}_{j+1} < \cdots < \hat{\sigma}_k < \hat{Q}\}$. There are two cases:

(a) The first case is when $s \leq j \leq k$. Then the sum of the weights of the chains $c_1^+$ and $c_2^+$ is given by

$$w(c_1^+) + w(c_2^+) = w_{[\emptyset, \hat{\sigma}]}(\hat{c}_1) \cdot a \cdot b \cdot w_{[\hat{\sigma}, \hat{Q}]}(\hat{c}_2).$$

Note that this case does not occur when $s = k + 1$.

(b) The second case is when $j = k + 1$. Then the sum of the weights of the chains is

$$w(c_1^+) + w(c_2^+) = w(\hat{c}) \cdot a.$$
(ii) Consider a chain \( c^- \) in the face poset of \( Q^- \) such that \( c^- \cap H \) and \( \hat{c} \) are the same. Then \( c^- \) is one of the following chains for some \( j \) with \( s-1 \leq j \leq k+1 \):

\[
\begin{align*}
c_1^- &= \{ \emptyset < \hat{\sigma}_1 < \ldots < \hat{\sigma}_j < \sigma_{j+1}^- < \ldots < \sigma_k^- = Q^- \}, \\
c_2^- &= \{ \emptyset < \hat{\sigma}_1 < \ldots < \hat{\sigma}_j < \sigma_j^- < \sigma_{j+1}^- < \ldots < \sigma_k^- = Q^- \}.
\end{align*}
\]

Let \( \sigma := \sigma_j \) and define chains \( \hat{c}_1 \) and \( \hat{c}_2 \) as in Case (i). If \( j \geq s \), then \( \text{relint} \sigma \cap (H^+ - H) \neq \emptyset \) and \( \text{relint} \sigma \cap (H^- - H) \neq \emptyset \). There are three cases:

(a) The first case is when \( s \leq j \leq k \). Then the sum of the weights of the chains \( c_1^- \) and \( c_2^- \) is given by

\[
w(c_1^-) + w(c_2^-) = w_{[\emptyset, \hat{\sigma}]}(\hat{c}_1) \cdot b \cdot a \cdot w_{[\hat{\sigma}, \hat{Q}]}(\hat{c}_2).
\]

Note that this case does not occur when \( s = k+1 \).

(b) The second case is when \( j = k+1 \). Then the only possible chain is \( c_2^- \) and its weight is

\[
w(c_2^-) = w(\hat{c}) \cdot b.
\]

(c) The third case is when \( j = s-1 \). Since \( \sigma_{s-1} \subset H \), we have \( \hat{\sigma}_{s-1} = \hat{\sigma}_{s-1}^- \).

Therefore the only possible chain is \( c_1^- \) and its weight is

\[
w(c_1^-) = w_{[\emptyset, \hat{\sigma}]}(\hat{c}_1) \cdot b \cdot (a - b) \cdot w_{[\hat{\sigma}, \hat{Q}]}(\hat{c}_2) = w(c).
\]

Now summing over all such chains in the face posets of \( Q^+ \) and \( Q^- \), we obtain

\[
\sum_{c^+} w(c^+) + \sum_{c^-} w(c^-) = \sum_{\hat{c}_1, \hat{c}_2} w_{[\emptyset, \hat{\sigma}]}(\hat{c}_1) \cdot d \cdot w_{[\hat{\sigma}, \hat{Q}]}(\hat{c}_2) + w(\hat{c}) \cdot c + w(c),
\]

where the first (resp. second) sum is over all chains \( c^+ \) (resp. \( c^- \)) in the face poset of \( Q^+ \) (resp. \( Q^- \)) such that \( c^+ \cap H = \hat{c} \) (resp. \( c^- \cap H = \hat{c} \)), and the third sum is over all pairs of chains \( \hat{c}_1, \hat{c}_2 \) obtained above such that \( \hat{c}_2 \) is nontrivial. Finally, summing over all chains in the face poset of \( Q \), we prove the theorem.

\[\square\]

Remark 4.4.3. The formula for the prism of a polytope in Proposition 4.4.1 is a special case of Theorem 4.4.2, since in this case

\[
Q = \text{Prism}(Q') \simeq Q' \times [0, 0.5] = Q^- \\
\simeq Q' \times [0.5, 1] = Q^+.
\]
and \( Q' \simeq Q' \times \{0.5\} = \hat{Q} \).

Also, the formula for the pyramid of a polytope is obtained from Theorem 4.4.2 by considering \( Q = \text{Bipyr}(Q') \) split by the hyperplane containing \( Q' \): in this case, \( Q^+ = Q^- = \text{Pyr}(Q') \) and there are no faces of \( \text{Bipyr}(Q') \) intersecting both open halfspaces nontrivially.

**Problem 4.4.4.** When \( Q(M_1) \cup Q(M_2) \) is a hyperplane split of \( Q(M) \) with a corresponding hyperplane \( H \), restate Theorem 4.4.2 in terms of matroids.

## 4.5 Rank 2 matroids

In this section we apply Theorem 4.3.1 and Theorem 4.4.2 to the \( \text{cd} \)-index of a matroid base polytope when a matroid has rank 2.

A (loopless) rank 2 matroid \( M \) on \([n]\) is determined up to isomorphism by the composition \( \alpha(M) \) of \([n]\) that gives the sizes \( \alpha_i \) of its parallelism classes.

Let \( \alpha := \alpha_1, \alpha_2, \ldots, \alpha_k \) be a composition of \( n \) with the length \( l(\alpha) = k \) and let \( M_\alpha \) be the corresponding rank 2 matroid on \([n]\). Note that

(i) \( M_\alpha \) is decomposable and connected if \( k > 3 \).

(ii) \( M_\alpha \) is indecomposable and connected if \( k = 3 \).

(iii) \( M_\alpha \) is indecomposable with two connected components if \( k = 2 \). Moreover, we have

\[
Q(M_\alpha) \cong Q(U_{1,\alpha_1}) \times Q(U_{1,\alpha_2}) \cong \Delta_{\alpha_1} \times \Delta_{\alpha_2},
\]

where \( U_{1,n} \) is the uniform matroid of rank 1 on \( n \) elements and \( \Delta_n \) is an \((n-1)\)-dimensional simplex.

For two weak compositions (i.e., compositions allowing 0 as parts) \( \alpha \) and \( \beta \) of the same length, we define \( \beta \leq \alpha \) if \( \beta_i \leq \alpha_i \) for all \( i = 1, 2, \ldots, l(\alpha) \). Let \( \bar{\beta} \) be the composition obtained from \( \beta \) by deleting 0 parts. If \( \alpha = (2, 4, 0, 6, 7) \) and \( \beta = (1, 3, 0, 6, 3) \), then \( \beta < \alpha \) and \( \bar{\beta} = (1, 3, 6, 3) \).

The following proposition expresses the \( \text{cd} \)-index of a matroid base polytope of a rank 2 matroid \( M \) with composition \( \alpha(M) = \alpha \) in terms of \( \text{cd} \)-indices of matroid
base polytopes of corresponding indecomposable matroids. For simplicity, we use the following notations:

\[
\lambda(\alpha, i) = \left( \sum_{j=1}^{i-1} \alpha_j, \alpha_i, \sum_{j=i+1}^{l(\alpha)} \alpha_j \right) \quad \text{for } 2 \leq i \leq l(\alpha) - 1,
\]

\[
\mu(\alpha, i) = \left( \sum_{j=1}^{i} \alpha_j, \sum_{j=i+1}^{l(\alpha)} \alpha_j \right) \quad \text{for } 1 \leq i \leq l(\alpha) - 1.
\]

If \(\alpha = (2, 4, 0, 6, 7)\), then \(\lambda(\alpha, 4) = (6, 6, 7)\) and \(\mu(\alpha, 4) = (12, 7)\).

**Proposition 4.5.1.** Let \(\alpha\) be a composition of \(n\) with at least three parts and \(M_\alpha\) be the corresponding rank 2 matroid on \([n]\). Then the cd-index of \(Q(M_\alpha)\) can be expressed as follows:

\[
\Psi(Q(M_\alpha)) = \sum_{i=2}^{l(\alpha)-1} \Psi(Q(M_{\lambda(\alpha, i)})) - \left( \sum_{i=2}^{l(\alpha)-2} \Psi(\Delta_{\mu(\alpha, i)} \times \Delta_{\mu(\alpha, i+2)}) \right) \cdot c
\]

\[
- \sum_{\beta < \alpha \atop l(\beta) \geq 4} \prod_{j=1}^{l(\beta)} \left( \sum_{i=2}^{l(\beta)-2} \Psi(\Delta_{\mu(\beta, i)} \times \Delta_{\mu(\beta, i+2)}) \right) \cdot d \cdot \Psi(\Delta_{n-|\beta|}).
\]

**Proof.** After the relabeling, one may assume that \(M_\alpha\) has parallelism classes \(P_1, \ldots, P_k\) such that \(\max P_i < \min P_{i+1}\) for all \(i = 1, \ldots, k - 1\) and \(|P_i| = \alpha_i\) for all \(i = 1, \ldots, k\). Define corresponding indecomposable matroids on \([n]\) as follows:

- \(M_3^i\) is the matroid with three parallelism classes \(\bigcup_{j=1}^{i-1} P_j, P_i, \text{ and } \bigcup_{j=i+1}^{k} P_j\).
- \(M_2^i\) is the matroid with two parallelism classes \(\bigcup_{j=1}^{i} P_j \text{ and } \bigcup_{j=i+1}^{k} P_j\).

Then \(M_3^i\) is isomorphic to \(M_{\lambda(\alpha, i)}\) and \(M_2^i\) is isomorphic to \(M_{\mu(\alpha, i)}\). For a subset \(S\) of \([n]\) such that \(M|_S\) is connected, one can also define \((M|_S)_1^3\) and \((M|_S)_1^2\) by considering \(M|_S\) as a matroid on \([n]\) such that the elements in \([n]\) \(S\) are loops.

We will use induction on \(k\).

**Base Case:** If \(k = 3\), then \(Q(M)\) is indecomposable and hence there is nothing to prove.

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Inductive Step: From Corollary 4.3.3, one can see that a hyperplane $H$ defined by $\sum_{e \in P_1 \cup \ldots \cup P_{k-2}} x_e = 1$ gives a hyperplane split $Q(M^+) \cup Q(M^-)$ of $Q(M)$ where $M^+$ is a matroid with parallelism classes $P_1, \ldots, P_{k-2}, P_{k-1} \cup P_k$ and $M^-$ has parallelism classes $P_1 \cup \cdots \cup P_{k-2}, P_{k-1}, P_k$. Also, $Q(M^+) \cap Q(M^-) = Q(M^0)$ where $M^0$ is a matroid with parallelism classes $P_1 \cup \cdots \cup P_{k-2}, P_{k-1}, P_k$. Note that there is no facet (i.e., facet corresponding to a flat of $M$) of $Q(M^0)$ which intersects both open halfspace given by $H$ nontrivially since every facet of $Q(M)$ corresponds to a base set of the form

$$\{ B \in \mathcal{B}(M) : |B \cap P_i| = 1 \}$$

for some $i$. (The corresponding facet has empty intersection with the open halfspace $H^- - H$ if $i \in [k-2]$ while it has empty intersection with $H^+ - H$ otherwise.)

Let $\sigma$ be the face of $Q(M)$ which has nonempty intersection with both open halfspaces given by $H$. Then $\sigma$ is the intersection of some facets of $Q(M)$ which are not facets. Also $[\sigma \cap H, Q(M^0)](\cong [\sigma, Q(M)])$ is the Boolean algebra of order $n - 1 - \dim \sigma$, and hence $\Psi(Q(M^0)/(\sigma \cap H)) = \Psi(\Delta_{n-|S|})$. Since each facet of $Q(M)$ which is not a facet corresponds to the deletion of an element of $[n]$, $\sigma$ corresponds to a matroid $M|_S$ for some subset $S$ of $[n]$. Also, $\sigma$ has nonempty intersection with both open halfspaces given by $H$ if and only if $M|_S$ has at least four parallelism classes, two of which are subsets of $P_{k-1}$ and $P_k$ respectively. Thus Theorem 4.4.2 implies

$$\Psi(Q(M)) = \Psi(Q(M^+)) + \Psi(Q(M^-)) - \Psi(Q(M^0)) \cdot c$$

$$- \sum_S \Psi(Q(M|_S)) \cdot d \cdot \Psi(\Delta_{n-|S|}),$$

where the sum in the second line runs over all proper subsets $S$ of $[n]$ such that $M|_S$ has at least four parallelism classes, two of which are subsets of $P_{k-1}$ and $P_k$ respectively. Since $M^+$ has $k - 1$ parallelism classes, the induction hypothesis implies

$$\Psi(Q(M)) = \sum_{i=2}^{k-1} \Psi(Q(M^2_i)) - \left( \sum_{i=2}^{k-2} \Psi(Q(M^2_i)) \right) \cdot c$$

$$- \sum_S \left( \sum_{k=2}^{p(M|_S)-2} \Psi(Q((M|_S)^2_i)) \right) \cdot d \cdot \Psi(\Delta_{n-|S|}),$$

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where the sum in the second line runs over all proper subsets $S$ of $[n]$ such that $M|_S$ has at least four parallelism classes, and $p(M|_S)$ is the number of parallelism classes of $M|_S$.

Since $M|_S$ corresponds to the composition

$$\beta_S := (|P_1 \cap S|, |P_2 \cap S|, \ldots, |P_k \cap S|) < \alpha$$

and there are

$$\prod_{j=1}^{k} \left( \frac{\alpha_j}{\beta_j} \right)$$

subsets corresponding to $\beta < \alpha$, we finish the proof.

Purtill [31] shows that the $\text{cd}$-index of the simplex $\Delta^n$ is the $(n + 1)$-st André polynomial. Using the formula for the $\text{cd}$-index of a product of two polytopes given by Ehrenborg and Readdy [16], one can calculate the second and the third terms in Proposition 4.5.1. We still don’t have a simple interpretation for the $\text{cd}$-index for $Q(M_\alpha)$ when $\alpha$ has three parts. The first few values of the $\text{cd}$-index for $Q(M_\alpha)$, where $\alpha$ has three parts, are displayed in Table 4.1.
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\Psi(Q(M_\alpha))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>$c^2 + d$</td>
</tr>
<tr>
<td>(2, 1, 1)</td>
<td>$c^3 + 3cd + 3dc$</td>
</tr>
<tr>
<td>(3, 1, 1)</td>
<td>$c^4 + 4c^2d + 8cdc + 5dc^2 + 7d^2$</td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>$c^4 + 5c^2d + 10cdc + 6dc^2 + 10d^2$</td>
</tr>
<tr>
<td>(4, 1, 1)</td>
<td>$c^5 + 5c^3d + 13c^2dc + 15cdc^2 + 20cd^2 + 7dc^3 + 18cdc + 22d^2c$</td>
</tr>
<tr>
<td>(3, 2, 1)</td>
<td>$c^5 + 6c^3d + 17c^2dc + 20cdc^2 + 28cd^2 + 9dc^3 + 26cdc + 33d^2c$</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>$c^5 + 7c^3d + 21c^2dc + 24cdc^2 + 36cd^2 + 10dc^3 + 34cdc + 42d^2c$</td>
</tr>
<tr>
<td>(5, 1, 1)</td>
<td>$c^6 + 6c^4d + 19c^3dc + 29c^2dc^2 + 38c^2d^2 + 24cdc^3 + 60cdc^3 + 72c^2d + 9d^4 + 33dc^2d + 65cdc^2 + 47d^2c^2 + 64d^3$</td>
</tr>
<tr>
<td>(4, 2, 1)</td>
<td>$c^6 + 7c^4d + 24c^3dc + 39c^2dc^2 + 52c^2d^2 + 33cdc^3 + 86cdc^3 + 104cd^2c + 12d^4 + 48dc^2d + 98dc^2d + 72d^2c^2 + 100d^3$</td>
</tr>
<tr>
<td>(3, 3, 1)</td>
<td>$c^6 + 7c^4d + 25c^3dc + 42c^2dc^2 + 55c^2d^2 + 36cdc^3 + 93cdc^3 + 114cd^2c + 13d^4 + 52dc^2d + 109dc^2d + 81d^2c^2 + 112d^3$</td>
</tr>
<tr>
<td>(3, 2, 2)</td>
<td>$c^6 + 8c^4d + 30c^3dc + 51c^2dc^2 + 69c^2d^2 + 42cdc^3 + 116cdc^3 + 142cd^2c + 14dc^2d + 64dc^2d + 136dc^2d + 98c^2d^2 + 142d^3$</td>
</tr>
</tbody>
</table>

Table 4.1: $cd$-index for $Q(M_\alpha)$ for composition $\alpha$ with three parts
Bibliography


