Topology of diagonal arrangements and flag enumerations of matroid base polytopes

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This thesis is dedicated to my parents and sister

Abstract

This thesis provides results on combinatorial properties of diagonal arragements and matroid base polytopes.

We study the topology of diagonal subspace arrangements in the first part. We prove that if a simplicial complex Δ is shellable, then the intersection lattice L_Δ for the corresponding diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres. Furthermore, we describe precisely the spheres in the wedge, based on the data of shelling. As a consequence, we give the homotopy type and the homology of the singularity link (and hence the homology of the complement) of diagonal subspace arrangement $\mathcal{A}_\Delta$. Also, we give some examples of diagonal arrangements $\mathcal{A}$ where the complement $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$, coming from rank 3 matroids.

In the second part, we study flag structures of matroid base polytopes. We describe faces of matroid base polytopes in terms of matroid data, and give conditions for hyperplane splits of matroid polytopes. Also, we apply this to the **cd**-index of a matroid base polytope of a rank 2 matroid.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

The objective of this work is to understand combinatorial properties of diagonal subspace arrangements and matroid base polytopes. More precisely, we study the topology of the intersection lattice of the diagonal subspace arrangement corresponding to a shellable simplicial complex. Also, we present some results about flags of faces in matroid base polytopes, and their **cd**-index.

This chapter provides informal summaries of Chapter 3 and 4. These summaries aim to be as non-technical as possible and precise definitions will be postponed until Chapter 2.

1.1 Topology of diagonal arrangements

In Chapter 3, we give the homotopy type of the intersection lattice of the diagonal arrangement corresponding to a shellable simplicial complex. More precisely, we show that this intersection lattice is homotopy equivalent to a wedge of spheres and describe the spheres in the wedge based on the data of shelling.

A *simplicial complex* on a vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ closed under taking subsets. The elements of a simplicial complex are called *faces* and maximal faces are called *facets*. A simplicial complex is *shellable* if there is a nice linear ordering, called a shelling, of facets (the formal definition will be given in Section 2.1). One of the nice properties of a shellable simplicial complex is that it is homotopy equivalent to a wedge of spheres. A shellable simplicial

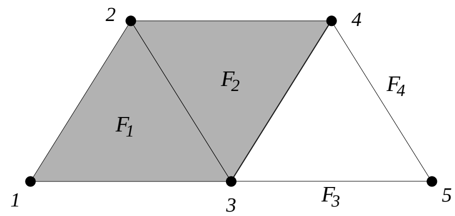


Figure 1.1: A shellable simplicial complex Δ

complex Δ on $[5] = \{1, 2, \ldots, 5\}$ with a shelling $F_1 = 123$ (short for $\{1, 2, 3\}$), $F_2 = 234$, $F_3 = 35$, $F_4 = 45$ is shown in Figure 1.1.

Consider $\mathbb{R}^n$ with coordinates $u_1, u_2, \ldots, u_n$. A *diagonal subspace* is a linear subspace of $\mathbb{R}^n$ of the form $u_{i_1} = \cdots = u_{i_r}$ with $r \geq 2$. A *diagonal arrangement* $\mathcal{A}$ is a finite set of diagonal subspaces of $\mathbb{R}^n$. There is a one-to-one correspondence between diagonal arrangements of $\mathbb{R}^n$ and simplicial complexes Δ on $[n] = \{1, 2, \ldots, n\}$ with $\dim \Delta \leq n - 3$ (see Section 3.1).

The simplest example of diagonal arrangements is the *thin diagonal*, which consists of one line $u_1 = \cdots = u_n$. The corresponding simplicial complex is the simplicial complex on $[n]$ whose unique face is the empty set. Another well-known example of diagonal arrangements is the *thick diagonal*, which is the collection of all subspaces $u_i = u_j$ for $1 \leq i < j \leq n$. The thick diagonal corresponds to the simplicial complex on $[n]$ which consists of all subsets of $[n]$ having less than $n-1$ elements.

Two important spaces associated with an arrangement $\mathcal{A}$ of linear subspaces in $\mathbb{R}^n$ are

$$\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H \quad \text{and} \quad \mathcal{V}^{\circ}_{\mathcal{A}} = \mathbb{S}^{n-1} \cap \bigcup_{H \in \mathcal{A}} H,$$

called the *complement* and the *singularity link*.

We are interested in the topology of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{V}^{\circ}_{\mathcal{A}}$ for a diagonal arrangement $\mathcal{A}$. In the mid 1980's Goresky and MacPherson [20] found a formula for the Betti numbers of $\mathcal{M}_{\mathcal{A}}$, while the homotopy type of $\mathcal{V}^{\circ}_{\mathcal{A}}$ was computed by Ziegler and Živaljević [39] (see Section 2.3). The answers are phrased in terms of the lower intervals in the *intersection lattice* $L_{\mathcal{A}}$ of the subspace arrangement $\mathcal{A}$, that is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion. For general subspace arrangements, these lower intervals in $L_{\mathcal{A}}$ can have arbitrary homotopy type (see [39, Corollary 3.1]). When we say that a finite lattice has some topological properties, such as shellability and homotopy type, it means the *order complex* of its proper part has those properties. The formal definition of lattices and order complexes will be given in Section 2.2.

Our goal is to find a general sufficient condition for the intersection lattice $L_{\mathcal{A}}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved. Suggested by a homological calculation (Theorem 3.5.3 below), we will prove the following main result (see Section 3.4).

Theorem 3.1.1. *Let Δ be a shellable simplicial complex. Then the intersection lattice L_Δ for the corresponding diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.*

The intersection lattice in Theorem 3.1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres (see Example 3.1.2).

Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. For the precise description, we introduce a notion of *shelling-trapped decomposition*, which is a decomposition of a subset of $[n]$ related to a shelling of Δ, in Section 3.1.

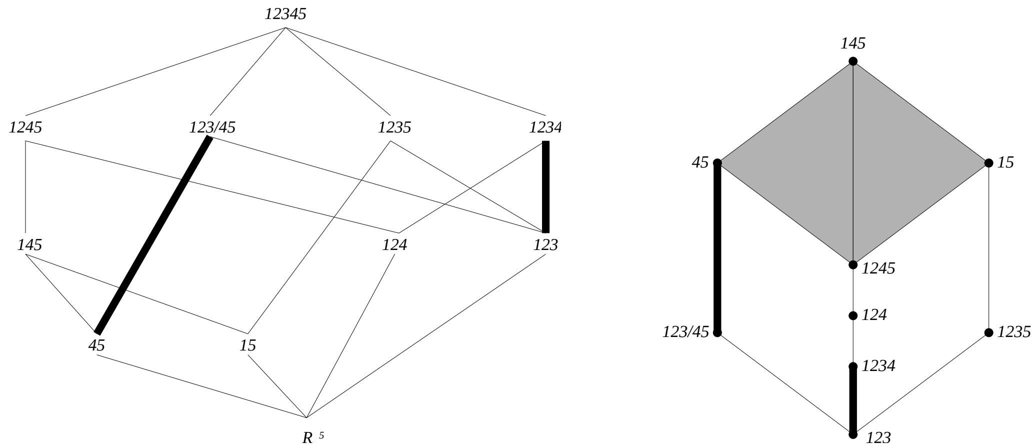
Let Δ be a shellable simplicial complex on $[n]$ and $\bar{\sigma}$ be the complement of the intersection of all facets of Δ. Then the wedge of spheres in Theorem 3.1.1 consists of $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition

$$D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$$

of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition D of $\bar{\sigma}$ and a permutation w of $[p-1]$, there exists a saturated chain $C_{D,w}$ (see Section 3.4) such that removing the simplices corresponding to these chains leaves a contractible simplicial complex. Note that the number and the dimension of spheres in a wedge are independent of the choice of shelling.



(a) L_Δ (b) The order complex of $\overline{L}_\Delta$

Figure 1.2: The intersection lattice L_Δ of $\mathcal{A}_\Delta$ and the order complex of $\overline{L}_\Delta$

Figure 1.2 shows the intersection lattice of the diagonal arrangement $\mathcal{A}_\Delta$ corresponding to the simplicial complex Δ in Figure 1.1 and the order complex for its proper part. The chains $C_{D,w}$ and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines. See Example 3.4.13 for more details.

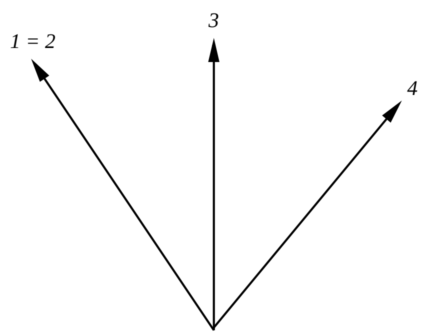
The homotopy type of the singularity link of $\mathcal{A}_\Delta$ for a shellable simplicial complex Δ is obtained from Theorem 3.1.1 and the result of Ziegler and Živaljević [39].

Corollary 3.4.14. *Let Δ be a shellable simplicial complex on $[n]$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension*

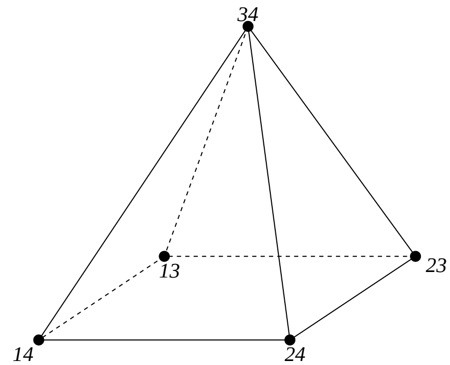
$$n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$$

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of some subset of $[n]$.

The homology version of this corollary (Theorem 3.5.3) can be proven without Theorem 3.1.1 by combining a result of Peeva, Reiner and Welker (Proposition 3.5.1) with results of Herzog, Reiner and Welker [22, Theorem 4, Theorem 9] along with the theory of *Golod rings* (see Section 3.5). It is what motivated us to



(a) $M_{2,1,1}$



(b) $Q(M_{2,1,1})$

Figure 1.3: A matroid $M_{2,1,1}$ represented as a vector configuration and its matroid base polytope whose vertices are labeled by bases of $M_{2,1,1}$

prove the stronger Corollary 3.4.14 and eventually Theorem 3.1.1.

1.2 Flag enumerations of matroid base polytopes

Chapter 4 is about flag structure and enumeration for matroid base polytopes. We associate a poset for each face of a matroid base polytope. We also give conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane and express the **cd**-index of matroid base polytopes of rank 2 matroids in terms of **cd**-indices of matroid base polytopes of corresponding indecomposable matroids.

A *matroid* is a combinatorial abstraction of vector configurations over any field, of hyperplane arrangements, and of graphs. For example, consider a collection of four vectors labeled $1, 2, 3$, and 4 in $\mathbb{R}^2$ where vectors 1 and 2 are parallel. This collection forms a matroid, denoted $M_{2,1,1}$, on a ground set $[4] = \{1, 2, 3, 4\}$ and its vector configuration is shown in Figure 1.3(a). The *rank* of a matroid M, denoted $r(M)$, is the maximal number of independent vectors. For example, the matroid $M_{2,1,1}$ has rank 2. The precise definition and some properties of matroids will be given in Section 2.4.

For a matroid M on a ground set $[n] = \{1, 2, \ldots, n\}$, a *matroid base polytope* $Q(M)$ is the polytope in $\mathbb{R}^n$ whose vertices are the incidence vectors of the bases of M. For the matroid $M_{2,1,1}$ on a ground set $[4] = \{1, 2, 3, 4\}$, its matroid base polytope is shown in Figure 1.3(b).

Since faces of a matroid base polytope $Q(M)$ are also matroid base polytopes, one can associate a matroid M_σ on $[n]$ for each face σ of $Q(M)$. Generalizing the characterization of facets of $Q(M)$ by Feichtner and Sturmfels [18, Proposition 2.6], we show that equivalence classes of factor-connected flags (defined in Section 4.2) of subsets of $[n]$ characterize faces of a matroid base polytope $Q(M)$ in Proposition 4.2.6. As a result, one can associate a poset for each face of $Q(M)$:

Theorem 4.1.1. *Let M be a matroid on a ground set $[n]$. For a face σ of the matroid base polytope $Q(M)$, one can associate a poset P_σ defined as follows:*

(i) the elements of P_σ are the connected components (defined in Section 4.2) of M_σ,

(ii) for distinct connected components C_1 and C_2 of M_σ, $C_1 < C_2$ if and only if

$$C_2 \subset S \subset [n] \text{ and } \sigma \subset H_S \text{ implies } C_1 \subset S,$$

where H_S is the hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = r(S)$, and $r(S)$ is the rank of S (defined in Section 2.4).

If σ is a vertex of a matroid base polytope, then this poset coincides with the poset defined by Billera, Jia and Reiner [4].

We found conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane.

Theorem 4.3.1. *Let M be a rank r matroid on $[n]$ and H be a hyperplane in $\mathbb{R}^n$ given by $\sum_{e \in S} x_e = k$. Then H decomposes $Q(M)$ into two matroid base polytopes if and only if*

(i) $r(S) \geq k$ and $r(S^c) \geq r - k$,

(ii) if I_1 and I_2 are k-element independent subsets of S such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$,

where $M|_S$ is the restriction of M on S and M/S is the contraction of M on S (see Section 2.4 for the precise definition).

The **cd**-index $\Psi(Q)$ of a polytope Q, a polynomial in the noncommutative variables $\mathbf{c}$ and $\mathbf{d}$, is an invariant that compactly encodes a large amount of enumerative information about flags of faces in Q. The **cd**-index will be defined in Section 4.4 for more general posets, namely Eulerian posets. Generalizing the formula of the **cd**-index of a prism and a pyramid of a polytope given by Ehrenborg and Readdy [16], we give the following theorem.

Theorem 4.4.2. *Let Q be a polytope in $\mathbb{R}^n$ and H be a hyperplane in $\mathbb{R}^n$. Then the following identity holds:*

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\widehat{Q}) \cdot \mathbf{c} - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot \mathbf{d} \cdot \Psi(\widehat{Q}/\hat{\sigma}),$$

where the sum is over all proper faces σ of Q intersecting both open halfspaces obtained by H nontrivially (notations will be defined in Section 2.4 and Section 4.4).

We apply Theorem 4.3.1 and Theorem 4.4.2 to express the **cd**-index of a matroid base polytope for a rank 2 matroid in terms of **cd**-indices of matroid base polytopes of corresponding indecomposable matroids (Proposition 4.5.1).

Chapter 2

Preliminaries

This chapter contains five sections, devoted respectively to simplicial complexes, posets and their order complexes, subspace arrangements, matroids, and convex polytopes. In each of these section, we provide the basic background materials which will be used throughout this thesis. Readers familiar with topics of a section may want to skip that section. We refer [28] for topological notions and [13] for notions on commutative algebra.

2.1 Simplicial complexes

This section contains basic information about shellable simplicial complexes.

An *(abstract) simplicial complex* Δ on a finite vertex set V is a collection of subsets of V satisfying

$$\tau \subset \sigma \in \Delta \text{ implies } \tau \in \Delta.$$

For $\sigma \in \Delta$, define $\dim \sigma = |\sigma| - 1$ and $\dim \Delta = \max_{\sigma\in\Delta} \dim \sigma$. The elements of Δ are called *faces* and the maximal faces are called *facets*. A simplicial complex Δ is *pure* if each facet has the same dimension.

The *geometric realization* $|\Delta|$ of the simplicial complex Δ is an embedding in which each vertex maps to a point and each (abstract) simplex maps to the (geometric) simplex spanned by the images of its vertices. Abusing notation, we sometimes identify a simplicial complex with its geometric realization.

For a simplicial complex Δ on the vertex set V, the *(Alexander) dual complex* Δ^* of Δ is defined to be

$$\Delta^* = \{\bar{\sigma} \subset V : \sigma = V - \bar{\sigma} \notin \Delta\}.$$

The *simplicial join* of two simplicial complex Δ and Δ' with disjoint vertex sets V and V' respectively, denoted by $\Delta * \Delta'$, is the simplicial complex on $V \cup V'$ with faces $\sigma \cup \tau$ where σ is a face of Δ and τ is a face of Δ'.

A simplicial complex is *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_q$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} 2^{F_i}) \cap 2^{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, q$, where $2^F = \{G | G \subseteq F\}$. Such an ordering of facets is called a *shelling order* or *shelling*. For further background on this notion, see [8], [9].

There are several equivalent definitions of shellability. The following restatement of shellability is often useful.

Lemma 2.1.1. ([8, Lemma 2.3]) *A linear order $F_1, F_2, \ldots, F_q$ of facets of a simplicial complex is a shelling if and only if for every F_i and F_k with $F_i < F_k$, there is a facet $F_j < F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k \lessdot F_k$, where $G \lessdot F$ means that G has codimension 1 in F.*

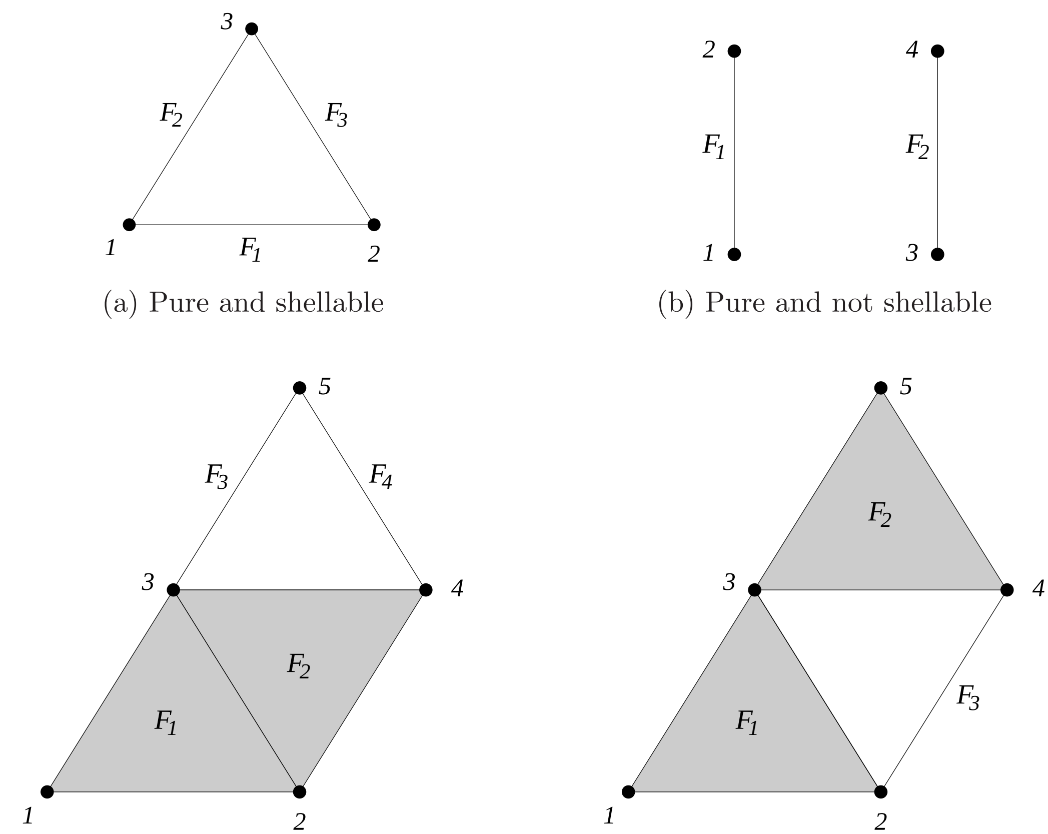
Example 2.1.2. The simplicial complexes in Figure 2.1(a) and (c) are shellable with shelling orders $F_1 = 12$, $F_2 = 13$, $F_3 = 23$ and $F_1 = 123$, $F_2 = 234$, $F_3 = 35$, $F_4 = 45$, respectively. On the other hand, since the intersection of two facets in Figure 2.1(b) is empty, i.e., has codimension two in any facet, the simplicial complex in Figure 2.1(b) is not shellable.

The *link* of a face σ of a simplicial complex Δ is

$$\mathrm{link}_\Delta \sigma (= \mathrm{link}\ \sigma) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}.$$

Björner and Wachs [9] show that the shellability is inherited by all links of faces in a simplicial complex.

Proposition 2.1.3. *If Δ is shellable, then so is $\mathrm{link}_\Delta \sigma$ for all faces $\sigma \in \Delta$, using the induced order on facets of $\mathrm{link}_\Delta \sigma$.*



(a) Pure and shellable (b) Pure and not shellable

(c) Nonpure and shellable (d) Nonpure and not shellable

Figure 2.1: Shellable and non-shellable simplicial complexes

Example 2.1.4. For the simplicial complex in Figure 2.1(d), one can see that link $\{3\}$ is not shellable since it is the simplicial complex in Figure 2.1(b) after relabeling. Thus Proposition 2.1.3 shows that the simplicial complex in Figure 2.1(d) is not shellable.

It is well-known that a pure d-dimensional shellable simplicial complex has the homotopy type of a wedge of d-spheres. Björner and Wachs [8] generalize this result to the nonpure case, i.e., a nonpure shellable simplicial complex has the homotopy type of a wedge of spheres, however these spheres need not be equidimensional.

2.2 Posets and their order complexes

This section contains background information about posets and their order complexes. The poset terminology and notations used in this thesis are standard and generally agree with [35], where proofs of the basic results in this section can be found.

A *partially ordered set* (or a *poset*) P is a set together with a reflexive, antisymmetric, transitive relation $\leq$ (or $\leq_P$ when there is a possibility of confusion). A *subposet* Q is a subset of P with the partial order induced on Q. If $x \leq y$, the *closed interval* $[x, y]$ is $\{z \in P : x \leq z \leq y\}$ and the *open interval* (x, y) is $\{z \in P : x < z < y\}$.

If $x, y \in P$, we say y *covers* x and write $x \lessdot y$ if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and y is drawn "above" x if $x < y$.

If there exists a unique minimal element in P, it is denoted $\hat{0}$. Similarly, if P has a unique maximal element, it is denoted $\hat{1}$. A poset P is *bounded* if it has both $\hat{0}$ and $\hat{1}$. A *chain* is a poset in which any two elements are comparable. A subset of a poset P is called a *chain* if it is a chain when regarded as a subposet of P. The chain C of P is called *saturated* if there does not exist $z \in P - C$ such that $x < z < y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is also a chain. A chain is *maximal* if it is not properly contained in any other chain. The *length* $l(C)$ of a finite chain C is defined by $l(C) = |C| - 1$. A poset P is *graded* (or *pure*) of rank n if every maximal chain of P has the same length n. In this case, there is a unique *rank function* ρ such that $\rho(x) = 0$ if x is minimal, and $\rho(y) = \rho(x) + 1$ if y covers x.

For $x, y \in P$, if there is a unique minimal element in $\{z \in P : z \geq x, z \geq y\}$, it is called the *join* of x and y and denoted $x \vee y$. Similarly, if there is a unique maximal element in $\{z \in P : z \leq x, z \leq y\}$, it is called the *meet* of x and y and denoted $x \wedge y$. A *lattice* is a poset L for which every pair of elements x and y has $x \vee y$ and $x \wedge y$.

The *order complex* $\Delta(P)$ of a poset P is the simplicial complex whose vertices

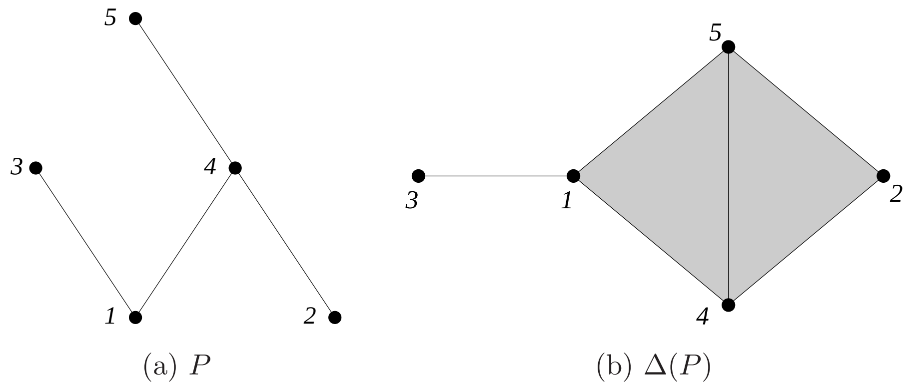


Figure 2.2: A poset P and its order complex $\Delta(P)$

are the elements of P and whose faces are the chains of P. Figure 2.2 is an example of a poset and its order complex. Note both P and $\Delta(P)$ in Figure 2.2 are nonpure.

For the order complex $\Delta((x, y))$ of an open interval (x, y), we will use the notation $\Delta(x, y)$. When we say that a finite lattice L has some topological properties, such as purity, shellability and homotopy type, it means the order complex of $\overline{L} = L - \{\hat{0}, \hat{1}\}$ has those properties.

2.3 Subspace arrangements

In this section, we give basic information about subspace arrangements.

Let $\mathbb{K}$ be a field. A *linear subspace* of $\mathbb{K}^n$ is a subspace of $\mathbb{K}^n$ containing $0 \in \mathbb{K}^n$, and an *affine subspace* of $\mathbb{K}^n$ is a translate of a linear space of $\mathbb{K}^n$. A *subspace arrangement* $\mathcal{A}$ in $\mathbb{K}^n$ is a finite collection of affine proper subspaces of $\mathbb{K}^n$. A subspace arrangement is called *central* if all subspaces are linear spaces. Throughout this thesis, we will take $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Consider $\mathbb{K}^n$ with coordinates $u_1, \ldots, u_n$. A *diagonal subspace* $U_{i_1 \cdots i_r}$ is a linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$ with $r \geq 2$. A *diagonal arrangement* (or a *hypergraph arrangement*) is a subspace arrangements consisting of diagonal subspaces.

The simplest example of diagonal arrangements is the *thin diagonal*, which consists of one line $u_1 = \cdots = u_n$. Another well-known example of diagonal arrangements is the *thick diagonal*, which is the collection of all subspaces $u_i = u_j$ for $1 \leq i < j \leq n$. This arrangement is also known as the *Braid arrangement* $\mathcal{B}_n$ and is well-studied (see [36] and Section 3.2 below). These two arrangements are particular cases of *k-equal arrangements* $\mathcal{A}_{n,k}$. The k-equal arrangement $\mathcal{A}_{n,k}$ in $\mathbb{K}^n$ is the arrangement of all subspaces $u_{i_1} = \cdots = u_{i_k}$ defined by setting k coordinates equal.

Two important spaces associated with a central arrangement $\mathcal{A}$ in $\mathbb{R}^n$ are

$$\mathcal{M}_{\mathcal{A}} = \mathbb{R}^n - \bigcup_{H\in\mathcal{A}} H \quad \text{and} \quad \mathcal{V}^{\circ}_{\mathcal{A}} = \mathbb{S}^{n-1} \cap \bigcup_{H\in\mathcal{A}} H,$$

called the *complement* and the *singularity link*. Note that $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{V}^{\circ}_{\mathcal{A}}$ are related by Alexander duality as follows:

$$H^i(\mathcal{M}_{\mathcal{A}}; \mathbb{F}) = H_{n-2-i}(\mathcal{V}^{\circ}_{\mathcal{A}}; \mathbb{F}) \qquad (\mathbb{F} \text{ is any field}).$$

In the mid 1980's Goresky and MacPherson [20] found a formula for the Betti numbers of $\mathcal{M}_{\mathcal{A}}$, i.e., the ranks of the singular cohomology groups $H^i(\mathcal{M}_{\mathcal{A}})$. The answer is phrased in terms of the lower intervals in the intersection lattice of the subspace arrangement $\mathcal{A}$. The *intersection lattice* $L_{\mathcal{A}}$ of a subspace arrangement $\mathcal{A}$ is the collection of all nonempty intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion.

Theorem 2.3.1. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$. Then*

$$\tilde{\beta}^i(\mathcal{M}_{\mathcal{A}}) = \sum_{x\in L_{\mathcal{A}}-\{\hat{0}\}} \tilde{\beta}_{codim(x)-2-i}(\hat{0}, x),$$

where the Betti number $\tilde{\beta}_d(P)$ of a poset P is the rank of the d-dimensional reduced simplicial homology group of the order complex $\Delta(P)$.

The homotopy type of $\mathcal{V}^{\circ}_{\mathcal{A}}$ is computed by Ziegler and Živaljević [39].

Theorem 2.3.2. *For every central subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$,*

$$\mathcal{V}^{\circ}_{\mathcal{A}} \simeq \bigvee_{x\in L_{\mathcal{A}}-\{\hat{0}\}} (\Delta(\hat{0}, x) * \mathbb{S}^{\dim(x)-1}),$$

where $\bigvee$ denotes wedge of spaces, "$*$" denotes join of spaces, and "$\simeq$" denotes homotopy equivalence.

These two results show that the topologies of $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{V}_{\mathcal{A}}^{\circ}$ reduce to topology of (lower) intervals in $L_{\mathcal{A}}$. For general subspace arrangements, these lower intervals in $L_{\mathcal{A}}$ can have arbitrary homotopy type (see [39, Corollary 3.1]).

2.4 Matroids

There are many equivalent ways to define a matroid (see [29], [32], [38] for more details). In this section, we provide two axiom systems for matroids.

A *matroid* M is an ordered pair $(E, \mathcal{I})$ where $E = E(M)$ is a finite set and $\mathcal{I} = \mathcal{I}(M)$ is a collection of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$,

(I2) If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$,

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

If M is the matroid $(E, \mathcal{I})$, M is called a matroid on E. The elements of $\mathcal{I}$ are called the *independent sets* of M. A maximal independent subset of M is called a *base* of M. All bases of M have the same size and the *rank* of M is the size of a base of M.

The next theorem shows that a matroid can be defined using bases.

Theorem 2.4.1 ([29, Corollary 1.2.5])**.** *A set $\mathcal{B}$ of subsets of E is the collection of bases of a matroid on E if and only if it satisfies the following conditions:*

(B1) $\mathcal{B} \neq \emptyset$,

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there is an element y of B_2 such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

For a matroid M on E and $e \in E$, e is an *isthmus* (or *coloop*) if e is contained in every base in $\mathcal{B}(M)$. If e is not an isthmus, the *deletion* $M \backslash e$ is a matroid on $E - \{e\}$ having independent sets

$$\mathcal{I}(M \backslash e) = \{I \in \mathcal{I}(M) : e \notin I\}$$

or bases

$$\mathcal{B}(M\backslash e) = \{B \in \mathcal{B}(M) : e \notin B\}.$$

For a subset $S = \{e_1, \ldots, e_k\}$ of E, one can define a *deletion*

$$M\backslash S := ((M\backslash e_1)\backslash e_2) \ldots \backslash e_k$$

and the *restriction*

$$M|_S := M\backslash(E - S).$$

The *rank* $r(S)$ of S is the rank of $M|_S$.

Given a matroid M on E and $e \in E$, one says that e is a *loop* if e lies in none of the bases in $\mathcal{B}(M)$. If e is not a loop, the *contraction* M/e is a matroid on $E - \{e\}$ having independent sets

$$\mathcal{I}(M/e) = \{I - \{e\} : I \in \mathcal{I}(M), e \in I\}$$

or bases

$$\mathcal{B}(M/e) = \{B - \{e\} : B \in \mathcal{B}(M), e \in B\}.$$

For a subset $S = \{e_1, \ldots, e_k\}$ of E, the contraction on S is defined by

$$M/S := ((M/e_1)/e_2) \ldots /e_k.$$

For a matroid M on E, its *dual* matroid $M^{\perp}$ is a matroid on E having bases

$$\mathcal{B}(M^{\perp}) = \{E - B : B \in \mathcal{B}(M)\}.$$

2.5 Convex polytopes

This section provides basic information about convex polytopes.

Recall that an *affine subspace* of $\mathbb{R}^n$ is a translate of linear subspace of $\mathbb{R}^n$. The *dimension* of an affine subspace is the dimension of the corresponding linear subspace. The *affine hull* of a set $S \subset \mathbb{R}^n$ is the intersection of all affine subspaces containing S.

A set $C \subset \mathbb{R}^n$ is *convex* if for any two points $x, y \in C$ it also contains the straight line segment with endpoints x and y. For any $S \subset \mathbb{R}^n$, the *convex* *hull* of S, denoted $\mathrm{conv}(S)$, is the intersection of all convex sets containing S. A *hyperplane* in $\mathbb{R}^n$ is the solution set of an equality $l(x) = c$ for some linear function l on $\mathbb{R}^n$ and some vector c of $\mathbb{R}^n$, and the corresponding *closed halfspace* is the solution set of an inequality $l(x) \geq c$. A *convex polytope* is the convex hull of a finite set of points. Equivalently, a convex polytope is a bounded set which is an intersection of finitely many closed halfspaces. The *dimension* of a convex polytope is the dimension of its affine hull.

A hyperplane $l(x) = c$ is a *supporting hyperplane* of a convex polytope Q if $l(x) \geq c$ for every point in Q. A *face* of Q is any intersection of Q with some supporting hyperplane. In particular, $\emptyset$ and Q itself are always faces of Q. Any face of Q is itself a convex polytope. The faces of dimension $0, 1$, and $\dim(Q) - 1$ are called *vertices, edges*, and *facets*, respectively. The *face lattice* of Q is a set of all faces of Q partially ordered by inclusion.

If Q is a d-dimensional polytope and x_0 is a point outside the affine hull of Q (for this we embed Q into $\mathbb{R}^n$ for some $n > d$), then the convex hull

$$\mathrm{Pyr}(Q) := \mathrm{conv}(Q \cup \{x_0\})$$

is a $(d+1)$-dimensional polytope called the *pyramid* over Q. Similarly, we construct the *bipyramid* over Q

$$\mathrm{Bipyr}(Q) := \mathrm{conv}(Q \cup \{x_+, x_-\})$$

by choosing two points x_+ and x_- outside the affine hull of Q such that an interior point of the segment $[x_+, x_-]$ is an interior point of Q.

For polytopes $Q_1 \subset \mathbb{R}^n$ and $Q_2 \subset \mathbb{R}^m$,

$$Q_1 \times Q_2 := \{(x, y) : x \in Q_1, y \in Q_2\}$$

is a polytope of dimension $\dim Q_1 + \dim Q_2$ and is called the *product* of Q_1 and Q_2. In particular, the *Prism* over a polytope Q is the product of Q with a line segment,

$$\mathrm{Prism}(Q) := Q \times \Delta_1,$$

where Δ_1 is a line segment (i.e., 1-dimensional polytope).

Let v be a vertex of a polytope Q and let $l(x) = c$ be a supporting hyperplane of Q defining v. The *vertex figure* Q/v of v is defined by

$$Q/v = Q \cap \{l(x) = c + \delta\}$$

where δ is an arbitrary small positive number. For a face σ of Q, the *face figure* Q/σ of σ is defined by

$$Q/\sigma = (\dots((Q/\sigma_0)/\sigma_1)\dots)/\sigma_k$$

where $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k = \sigma$ is a maximal chain with $\dim \sigma_i = i$. For faces σ and τ of Q with $\sigma \subset \tau$, the face lattice of the face figure τ/σ is the interval $[\sigma, \tau]$.

Chapter 3

Topology of Diagonal Arrangements

3.1 Main results

For a simplicial complex Δ on a vertex set $[n]$ with $\dim \Delta \leq n-3$, one can associate a diagonal arrangement $\mathcal{A}_\Delta$ as follows. For a facet F of Δ, let $U_{\overline{F}}$ be the diagonal subspace $u_{i_1} = \cdots = u_{i_r}$ where $\overline{F} = [n] - F = \{i_1, \ldots, i_r\}$. Define

$$\mathcal{A}_\Delta = \{U_{\overline{F}} | F \text{ is a facet of } \Delta\}.$$

For each diagonal arrangement $\mathcal{A}$, one can find a simplicial complex Δ such that $\mathcal{A} = \mathcal{A}_\Delta$.

Our goal is to find a general sufficient condition for the intersection lattice $L_\mathcal{A}$ of a diagonal arrangement $\mathcal{A}$ to be well-behaved. Björner and Welker [10] show that $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres, where $\mathcal{A}_{n,k}$ is the k-equal arrangement in $\mathbb{R}^n$ (see Section 3.3). More generally, Kozlov [25] shows that $L_\mathcal{A}$ is shellable if $\mathcal{A}$ satisfies certain technical conditions (see Section 3.3). Suggested by a homological calculation (Theorem 3.5.3 below), we will prove the following main result, capturing the homotopy type assertion from [25] (see Section 3.4).

Theorem 3.1.1. *Let Δ be a shellable simplicial complex. Then the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of spheres.*

Furthermore, one can describe precisely the spheres in the wedge, based on the shelling data. Let Δ be a simplicial complex on $[n]$ with a shelling order $F_1, \dots, F_q$ on its facets. Let σ be the intersection of all facets, and $\bar{\sigma}$ its complement. Let $G_1 = F_1$ and for each $i \geq 2$ let G_i be the face of F_i obtained by intersecting the walls of F_i that lie in the subcomplex generated by $F_1, \dots, F_{i-1}$, where a *wall of* F_i is a codimension 1 face of F_i. An (*unordered*) *shelling-trapped decomposition of* $\bar{\sigma}$ (*over* Δ) is defined to be a family $\{(\bar{\sigma}_1, F_{i_1}), \dots, (\bar{\sigma}_p, F_{i_p})\}$ such that $\{\bar{\sigma}_1, \dots, \bar{\sigma}_p\}$ is a decomposition of $\bar{\sigma}$ as a disjoint union

$$\bar{\sigma} = \bigsqcup_{j=1}^{p} \bar{\sigma}_j$$

and $F_{i_1}, \dots, F_{i_p}$ are facets of Δ such that $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all j. Then the wedge of spheres in Theorem 3.1.1 consists of $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \dots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$. Moreover, for each shelling-trapped decomposition D of $\bar{\sigma}$ and a permutation w of $[p-1]$, there exists a saturated chain $C_{D,w}$ (see Section 3.4) such that removing the simplices corresponding to these chains leaves a contractible simplicial complex.

The following example shows that the intersection lattice in Theorem 3.1.1 is not shellable in general, even though it has the homotopy type of a wedge of spheres.

Example 3.1.2. Let Δ be a simplicial complex on $[8] = \{1, 2, \dots, 8\}$ with a shelling $123456, 127, 237, 137, 458, 568, 468$. Then $\Delta(U_{78}, \hat{1})$ is a disjoint union of two circles, hence is not shellable. Therefore, the intersection lattice L_Δ for the diagonal arrangement $\mathcal{A}_\Delta$ is also not shellable. The intersection lattice L_Δ is shown in Figure 3.1 (thick lines represent the open interval $(U_{78}, \hat{1})$). In Figures, the subspace $U_{i_1 \dots i_r}$ is labeled by $i_1 \dots i_r$. Also note that a facet F of Δ corresponds to the subspace $U_{[n]-F}$. For example, the facet 127 corresponds to U_{34568}.

The next example shows that there are nonshellable simplicial complexes whose intersection lattices are shellable.

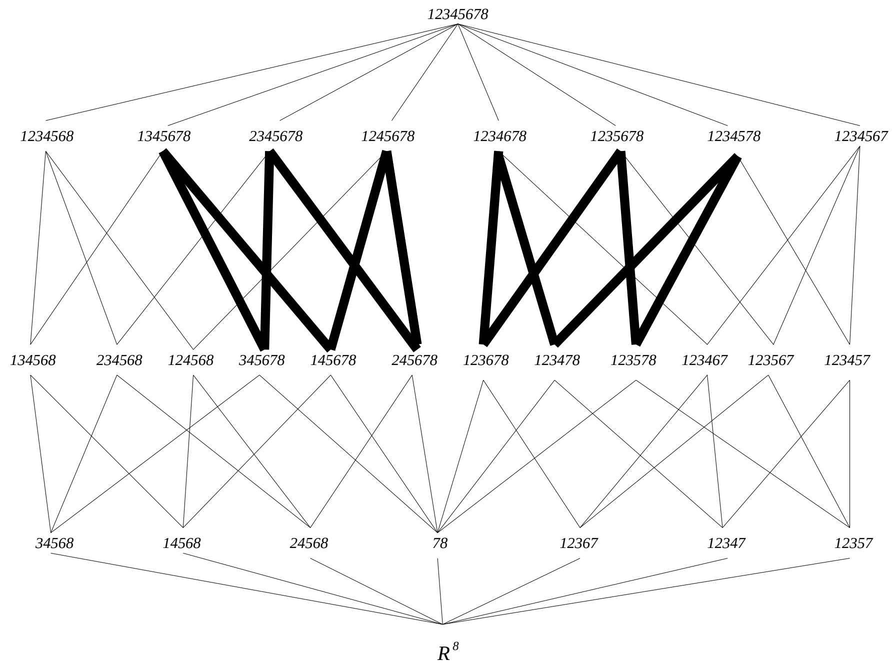


Figure 3.1: Nonshellable intersection lattice L_Δ for shellable Δ ($U_{i_1 \dots i_r}$ is labeled by $i_1 \dots i_r$.)

Example 3.1.3. Let Δ be a simplicial complex on $\{1, 2, 3, 4\}$ whose facets are 12 and 34. Then Δ is not shellable as we have seen in Example 2.1.2. But the order complex of $\overline{L}_\Delta$ consists of two vertices, hence is shellable.

This chapter is organized as follows: Section 3.2 contains applications to group cohomology. In Section 3.3, we give Kozlov's result and show that its homotopy type consequence is a special case of Theorem 3.1.1. Also, we give a new proof of Björner and Welker's result using Theorem 3.1.1. Section 3.4 gives a proof of Theorem 3.1.1, and we deduce the homotopy type of the singularity link of $\mathcal{A}_\Delta$ for a shellable simplicial complex Δ. In Section 3.5, we give the homology of the singularity link (and hence the homology of the complement) of a diagonal arrangement $\mathcal{A}_\Delta$ for a shellable simplicial complex Δ without using Theorem 3.1.1. In Section 3.6, we give some examples in which $\mathcal{M}_\mathcal{A}$ are $K(\pi, 1)$, coming from matroids.

3.2 Motivation from group cohomology

In this section, we give an application of topology of diagonal arrangements to group cohomology. Recall that an Eilenberg-MacLane space (or a $K(\pi, n)$-space) is a connected cell complex with all homotopy groups except the n-th homotopy group being trivial and the n-th homotopy group isomorphic to π.

Let the CW complex X be a $K(\pi, 1)$-space. Let $\widetilde{X}$ be its universal cover, i.e., the unique simply connected cover. Let $p : \widetilde{X} \to X$ be the covering map. Then we have

(i) $\widetilde{X}$ is also a CW complex.

(ii) $\pi_1(X) = \pi$ acts freely on $\widetilde{X}$. Explicitly, the open cells of $\widetilde{X}$ lying over an open cell σ of X are simply the connected components of $p^{-1}\sigma$. These cells are permuted freely and transitively by π, and each is mapped homeomorphically onto σ under p. Thus $C_*(\widetilde{X})$ is a complex of free $\mathbb{Z}\pi$-modules with one basis element for each cell of X.

(iii) $\pi_1(\widetilde{X}) = 1$ and $\pi_i(\widetilde{X}) = \pi_i(X) = 0$ for $i > 1$. Hence $\widetilde{X}$ is contractible.

Thus the augmented cellular complex of $\widetilde{X}$

$$\begin{array}{ccccccccccc} \cdots & \to & C_2(\widetilde{X}) & \to & C_1(\widetilde{X}) & \to & C_0(\widetilde{X}) & \to & C_{-1}(\widetilde{X}) & \to & 0 \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ \cdots & \to & (\mathbb{Z}\pi)^{\beta_2} & \to & (\mathbb{Z}\pi)^{\beta_1} & \to & (\mathbb{Z}\pi)^{\beta_0} & \to & \mathbb{Z} & \to & 0, \end{array}$$

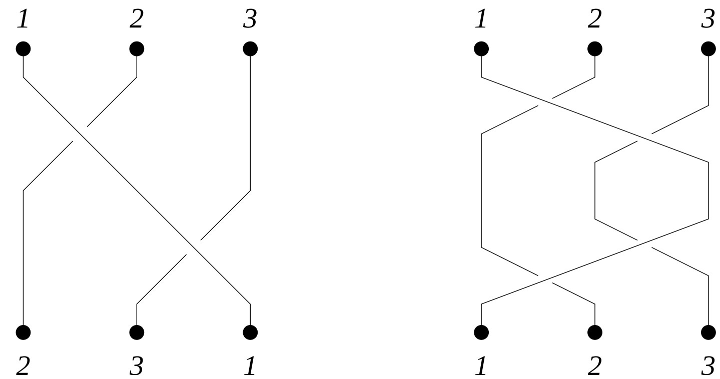
where β_i is the number of i-dimensional cells of X, is exact. Hence removing $C_{-1}(\widetilde{X})$ (i.e., taking the usual cellular complex of $\widetilde{X}$) gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ (see [11, Proposition 4.1]).

Tensoring with $\mathbb{Z}$ over $\mathbb{Z}\pi$ and taking homology gives $\mathrm{Tor}_n^{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z})$. On the other hand, the same evaluation gives $H_n(X; \mathbb{Z})$ since

$$C_i(\widetilde{X}) \otimes_{\mathbb{Z}\pi} \mathbb{Z} = (\mathbb{Z}\pi)^{\beta_i} \otimes_{\mathbb{Z}\pi} \mathbb{Z} = \mathbb{Z}^{\beta_i} = C_i(X; \mathbb{Z}) \quad \text{for all } i.$$

Hence

$$\mathrm{Tor}_n^{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z}) = H_n(X; \mathbb{Z}).$$



(a) A braid (b) A pure braid

Figure 3.2: Braids on three strands

Similarly, one can get

$$\mathrm{Ext}^n_{\mathbb{Z}\pi}(\mathbb{Z}, \mathbb{Z}) = H^n(X; \mathbb{Z})$$

by applying the functor $\mathrm{Hom}_{\mathbb{Z}\pi}(-, \mathbb{Z})$ and taking cohomology. Therefore, one can evaluate Tor and Ext of $\mathbb{Z}$ in terms of the integral (co)homology of the $K(\pi, 1)$-space X.

Now, we give two examples of diagonal arrangement whose complement is a $K(\pi, 1)$-space.

Let $B(n)$ be the *braid group* on n strands. Figure 3.2(a) shows a braid on three strands. There is a natural surjection $B(n) \to \mathfrak{S}_n$ which sends each braid to the permutation of its ends. The image of the braid in Figure 3.2(a) is the 3-cycle $(1, 3, 2)$. The kernel of this map is called the *pure braid group* $PB(n)$. The corresponding pure braids have the property that each strand returns to its point of origin. Figure 3.2(b) shows a pure braid on three strands.

Consider a pure braid on n strands as a set of n non-intersecting arcs in $\mathbb{R}^2 \times [0, 1]$ such that

(i) arcs connect $(1, 0, 0), \ldots, (n, 0, 0)$ with $(1, 0, 1), \ldots,$ $(n, 0, 1)$ in this order, and

(ii) arcs descend monotonically.

Define a path in $\mathbb{C}^n$ by

$$(a_1(t) + ib_1(t), \ldots, a_n(t) + ib_n(t)), \qquad 0 \leq t \leq 1,$$

where $(a_j(t), b_j(t), t)$ is on the arc connecting $(j, 0, 0)$ and $(j, 0, 1)$ for all $1 \leq j \leq n$.

Since $a_j(t) + ib_j(t) \neq a_k(t) + ib_k(t)$ whenever $j \neq k$, and $a_j(0) = a_j(1) = j$ and $b_j(0) = b_j(1) = 0$ for all $1 \leq j \leq n$, we get a closed path in $\mathcal{M}_{\mathcal{B}_n}$ with the base point $(1, 2, \ldots, n)$.

By using this correspondence, one can show the second part of the following theorem given by Fadell and Neuwirth [17].

Theorem 3.2.1. *Let $\mathcal{B}_n$ be the braid arrangement in $\mathbb{C}^n$. Then*

(i) $\mathcal{M}_{\mathcal{B}_n}$ is $K(\pi, 1)$ space, and

(ii) its fundamental group is isomorphic to the pure braid group $PB(n)$.

Khovanov [23] gives a real counterpart of Theorem 3.2.1. For that result, we need to define the *twin group*. Let $\mathcal{F}_n$ be the topological space of configurations of n continuous arcs in the strip $\mathbb{R} \times [0, 1]$ such that

(i) arcs connect points $(1, 1), \ldots, (n, 1)$ with $(1, 0), \ldots, (n, 0)$ in some order,

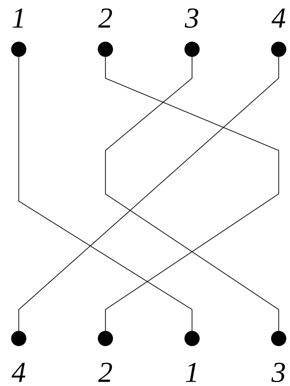
(ii) arcs descend monotonically, and

(iii) no three arcs have a common point.

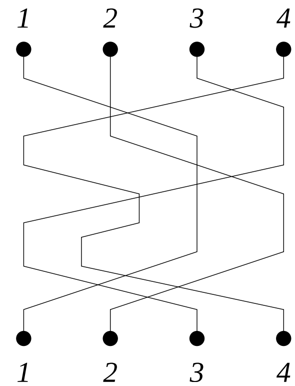
Two elements $a, b \in \mathcal{F}_n$ are multiplied by putting one on top of the other and squeezing the interval $[0, 2]$ to $[0, 1]$. Define a *twin* on n arcs to be a connected component of the space $\mathcal{F}_n$. Then twins form a group and we call it the *twin group on n arcs*. There is a natural surjection of the twin group to the symmetric group $\mathfrak{S}_n$ given by sending each twin to the permutation of its ends. The kernel of this surjection is called the *pure twin group on n arcs*. An example of a twin and a pure twin on four arcs is shown in Figure 3.3.

For each pure twin on n arcs, we can define the closed path in $\mathbb{R}^n$ based on $(1, 2, \ldots, n)$ given by

$$(a_1(t), \ldots, a_n(t)), \qquad 0 \leq t \leq 1$$



(a) A twin



(b) A pure twin

Figure 3.3: Twins on four arcs

where $(a_i(t), t)$ lies on the arc that connects $(i, 0)$ and $(i, 1)$. Since no three arcs have a common point, this closed path lies in $\mathcal{M}_{\mathcal{A}_{n,3}}$. This explains the second part of the following theorem given by Khovanov [23].

Theorem 3.2.2. *Let $\mathcal{A}_{n,3}$ be the 3-equal arrangement in $\mathbb{R}^n$. Then*

(i) $\mathcal{M}_{\mathcal{A}_{n,3}}$ is a $K(\pi, 1)$ space, and

(ii) the fundamental group of $\mathcal{M}_{\mathcal{A}_{n,3}}$ is isomorphic to the pure twin group on n arcs.

3.3 Special cases that were known

In this section, we give Kozlov's theorem and show how its consequence for homotopy type follows from Theorem 3.1.1. Also, we give a new proof of Björner and Welker's theorem about the intersection lattice of the k-equal arrangements using Theorem 3.1.1.

Kozlov [25] shows that $\mathcal{A}_\Delta$ has a shellable intersection lattice if Δ satisfies some conditions. This class includes k-equal arrangements and all other diagonal arrangements for which the intersection lattice was proved shellable.

Theorem 3.3.1. ([25, Corollary 3.2]) *Consider a partition of*

$$[n] = E_1 \sqcup \cdots \sqcup E_r$$

such that $\max E_i < \min E_{i+1}$ *for* $i = 1, \ldots, r-1$. *Let*

$$f : \{1, 2, \ldots, r\} \to \{2, 3, \ldots\}$$

be a nondecreasing map. Let Δ *be a simplicial complex on* $[n]$ *such that* F *is a facet of* Δ *if and only if*

(i) $|E_i - F| \leq 1$ *for* $i = 1, \ldots, r$;

(ii) *if* $\min \overline{F} \in E_i$ *then* $|F| = n - f(i)$.

Then the intersection lattice for $\mathcal{A}_\Delta$ *is shellable. In particular,* $L_{\mathcal{A}_\Delta}$ *has the homotopy type of a wedge of spheres.*

Proposition 3.3.2. Δ *in Theorem 3.3.1 is shellable.*

Proof. We claim that a shelling order for Δ is $F_1, F_2, \ldots, F_q$ such that the words $w_1, w_2, \ldots, w_q$ are in lexicographic order, where w_i is the increasing array of elements in $\overline{F}_i$. Let F_s, F_t be two facets of Δ with $1 \leq s < t \leq q$. Then $w_s \prec_{lex} w_t$. Let m be the first number in $[r]$ such that $E_m - F_s \neq E_m - F_t$. Construct the word w as follows:

(i) $w \cap E_i = w_s \cap E_i$ for $i = 1, \ldots, m$;

(ii) for $i = m+1, \ldots, q$,

$$w \cap E_i = \begin{cases} w_t \cap E_i & \text{if } |w \cap (\cup_{j=1}^{i} E_j)| \leq f(l); \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\min w_s \in E_l$.

Note that the length of w is $f(l)$ and $w \prec_{lex} w_t$. Let F be a set of all elements which do not appear in w. Since F satisfies the two conditions in Theorem 3.3.1, F is a facet of Δ. Since $F \cap F_t = F_t - (E_m - F_s)$ and $E_m - F_s$ is a subset of F_t of size 1, $F \cap F_t$ has codimension 1 in F_t. Also $F_s \cap F_t \subseteq F \cap F_t$. Hence $F_1, F_2, \ldots, F_q$ is a shelling by Lemma 2.1.1. □

$\min \overline{F}$	F	w	$\min \overline{F}$	F	w	$\min \overline{F}$	F	w
1	23456	17	2	1356	247	3	1256	347
	23457	16		1357	246		1257	346
	23467	15		1367	245		1267	345
	23567	14						
	24567	13						
	34567	12						

Table 3.1: Facets of Δ and corresponding words

Example 3.3.3. Consider the partition of

$$[7] = \{1\} \sqcup \{2,3\} \sqcup \{4\} \sqcup \{5,6,7\}$$

and the function f given by $f(1) = 2, f(2) = 3, f(3) = 4$, and $f(4) = 5$. Then the facets of the simplicial complex that satisfy the conditions of Corollary 3.3.1 and the corresponding words are given in Table 3.1. Thus the ordering 34567, 24567, 23567, 23467, 23457, 23456, 1367, 1357, 1356, 1267, 1257 and 1256 is a shelling for this simplicial complex.

One can also use Theorem 3.1.1 to recover the following theorem of Björner and Welker [10].

Theorem 3.3.4. *The order complex of the intersection lattice $L_{\mathcal{A}_{n,k}}$ has the homotopy type of a wedge of spheres consisting of*

$$(p-1)! \sum_{0=i_0 \leq i_1 \leq \cdots \leq i_p = n-pk} \prod_{j=0}^{p-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}$$

copies of $(n-3-p(k-2))$-dimensional spheres for $1 \leq p \leq \lfloor \frac{n}{k} \rfloor$.

Proof. It is clear that $\mathcal{A}_{n,k} = \mathcal{A}_{\Delta_{n,n-k}}$, where $\Delta_{n,n-k}$ is a simplicial complex on $[n]$ whose facets are all $n-k$ subsets of $[n]$. Here, $\sigma = \emptyset$, $\bar{\sigma} = [n]$. By ordering the elements of each facet in increasing order, the lexicographic order of facets of $\Delta_{n,n-k}$ gives a shelling. Also, one can see that the facet of the form

$F_i = \{1, 2, \ldots, m, a_{m+1}, \ldots, a_{n-k}\}$, where $m + 1 < a_{m+1} < \cdots < a_{n-k}$, has $G_i = \{1, \ldots, m\}$. Thus, $G_i \subseteq \sigma_i \subseteq F_i$ implies $\overline{F}_i \subseteq \bar{\sigma}_i \subseteq \overline{G}_i = \{m+1, \ldots, n\}$. Note that $\min \bar{\sigma} = \min \overline{F} = m+1$ and $\overline{F}$ has k elements. Thus, in any shelling-trapped decomposition $[n] = \sqcup_{j=1}^{p} \bar{\sigma}_j$, one has $p \leq \lfloor \frac{n}{k} \rfloor$.

Let $1 \leq p \leq \lfloor \frac{n}{k} \rfloor$ and $0 = i_0 \leq i_1 \leq \cdots \leq i_p = n - pk$. We will construct a shelling-trapped family $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ as in Theorem 3.1.1. Because $F_{i_1} < \cdots < F_{i_p}$, we have $\min \bar{\sigma}_1 > \cdots > \min \bar{\sigma}_p$. In particular, $1 \in \overline{F}_{i_p} \subseteq \bar{\sigma}_p$. Thus there are $\binom{n-1}{k-1}$ ways to pick $\overline{F}_{i_p}$ (equivalently, F_{i_p}). Now suppose that we have chosen $F_{i_p}, \ldots, F_{i_{p-j+1}}$. We pick $F_{i_{p-j}}$ so that $\min \overline{F}_{i_{p-j}} = \min \bar{\sigma}_{i_{p-j}}$ is the $i_j + 1$st element in $[n] - (F_{i_p} \cup \cdots \cup F_{i_{p-j+1}})$. Then we have $\binom{n-jk-i_j-1}{k-1}$ ways to choose $F_{i_{p-j}}$. For each $j = 1, \ldots, p$, there are $i_j - i_{j-1}$ elements in

$$[n] - (F_{i_p} \cup \cdots \cup F_{i_{p-j+1}})$$

which are strictly between $\min \overline{F}_{i_{p-j+1}}$ and $\min \overline{F}_{i_{p-j}}$ and they must be contained in one of $\bar{\sigma}_p, \ldots, \bar{\sigma}_{p-j+1}$ (i.e., there are $j^{i_j - i_{j-1}}$ choices). Therefore there are

$$\prod_{j=0}^{p-1} \binom{n - jk - i_j - 1}{k - 1} \prod_{j=1}^{p} j^{i_j - i_{j-1}} = \prod_{j=0}^{p-1} \binom{n - jk - i_j - 1}{k - 1} (j+1)^{i_{j+1} - i_j}$$

shelling-trapped families. By Theorem 3.1.1, each of those families contributes $(p-1)!$ copies of spheres of dimension

$$p(2-n) + \sum_{j=1}^{p} (n-k) + n - 3 = n - 3 - p(k-2).$$

□

3.4 Proof of main theorem

Theorem 3.1.1 will be deduced from a more general statement about homotopy types of lower intervals in $L_{\mathcal{A}}$, Theorem 3.4.1 below. Throughout this section, we assume that Δ is a simplicial complex on $[n]$ with $\dim \Delta \leq n - 3$.

Theorem 3.4.1. *Let $F_1, \ldots, F_q$ be a shelling of Δ and $U_{\bar{\sigma}}$ a subspace in L_Δ for some subset $\bar{\sigma}$ of $[n]$. Then $\Delta(\hat{0}, U_{\bar{\sigma}})$ is homotopy equivalent to a wedge of spheres, consisting of $(p-1)!$ copies of spheres of dimension*

$$\delta(D) := p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3$$

for each shelling-trapped decomposition $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of $\bar{\sigma}$.

Moreover, for each such shelling-trapped decomposition D and each permutation w of $[p-1]$, one can construct a saturated chain $C_{D,w}$ (see Section 3.4.1 below), such that if one removes the corresponding $\delta(D)$-dimensional simplices for all pairs (D, w), the remaining simplicial complex $\widehat{\Delta}(\hat{0}, U_{\bar{\sigma}})$ is contractible.

To prove this result, we begin with some preparatory lemmas.

First of all, one can characterize exactly which subspaces lie in L_Δ when Δ is shellable. Recall that for $\bar{\sigma} = \{i_1, \ldots, i_r\} \subseteq [n]$, we denote by $U_{\bar{\sigma}}$ the linear subspace of the form $u_{i_1} = \cdots = u_{i_r}$. We also use the notation $U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$ to denote $U_{\bar{\sigma}_1} \cap \cdots \cap U_{\bar{\sigma}_p}$ for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$.

A simplicial complex is called *gallery-connected* if any pair F, F' of facets are connected by a path

$$F = F_0, F_1, \ldots, F_{r-1}, F_r = F'$$

of facets such that $F_i \cap F_{i-1}$ has codimension 1 in F_i for $i = 1, \ldots, r$. Since it is known that Cohen-Macaulay simplicial complexes are gallery-connected, shellable simplicial complexes are gallery-connected.

Lemma 3.4.2. *(i) Given any simplicial complex Δ on $[n]$, every subspace H in L_Δ has the form*

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$ such that σ_i can be expressed as an intersection of facets of Δ for $i = 1, 2, \ldots, p$.

(ii) Conversely, when Δ is gallery-connected, every subspace H of $\mathbb{R}^n$ that has the above form lies in L_Δ.

Proof. To see (i), note that every subspace H in L_Δ has the form

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$. Since $H = \bigvee_{F \in \mathcal{F}} U_{\overline{F}}$ for some family $\mathcal{F}$ of facets of Δ,

$$U_{\bar{\sigma}_j} = \bigvee_{F \in \mathcal{F}_j} U_{\overline{F}}$$

for some subfamily $\mathcal{F}_j$ of $\mathcal{F}$ for all $j = 1, \ldots, p$. Therefore

$$\sigma_j = \bigcap_{F \in \mathcal{F}_j} F$$

for $j = 1, \ldots, p$.

For (ii), suppose that H has the form

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$ such that σ_i can be expressed as an intersection of facets of Δ for $i = 1, 2, \ldots, p$. It is enough to show the case when $H = U_{\bar{\sigma}}$. Since gallery-connectedness is inherited by all links of faces in a simplicial complex, we may assume $\sigma = \bigcap_{F \in \mathcal{F}} F$, where $\mathcal{F}$ is the set of all facets of Δ, without loss of generality. Then $\bar{\sigma} = \bigcup_{F \in \mathcal{F}} \overline{F}$.

We claim that the simplicial complex Γ whose facets are $\{\overline{F} \mid F \in \mathcal{F}\}$ is connected. Since $\dim \Delta \leq n - 3$, every facet of Γ has at least two elements. Let $\overline{F}, \overline{F'}$ be two facets of Γ with $F < F'$. Since Δ is gallery-connected, there is $F = F_1, F_2, \ldots, F_k = F'$ such that $F_i \cap F_{i-1}$ has codimension 1 in F_i for all $i = 2, \ldots, k$. Thus $\overline{F}_i$ and $\overline{F}_{i-1}$ share at least one vertex for all $i = 2, \ldots, k$. This implies that $\overline{F}$ and $\overline{F'}$ are connected. Hence Γ is connected.

Therefore $\bar{\sigma} = \bigcup_{F \in \mathcal{F}} \overline{F} = \bigvee_{F \in \mathcal{F}} \overline{F}$. □

The next example shows that the conclusion of Lemma 3.4.2(ii) can fail when Δ is not assumed to be gallery-connected.

Example 3.4.3. Let Δ be a simplicial complex with two facets 123 and 345. Then Δ is not gallery-connected. Since L_Δ has only three subspaces U_{12}, U_{45} and $U_{12/45}$, it does not have the subspace U_{1245}, even though $\overline{1245} = 3$ is an intersection of facets 123 and 345 of Δ. Thus the conclusion of Lemma 3.4.2(ii) fails for Δ.

The following easy lemma shows that every lower interval $[\hat{0}, H]$ can be written as a product of lower intervals of the form $[\hat{0}, U_{\bar{\sigma}}]$.

Lemma 3.4.4. *Let $H \in L_\Delta$ be a subspace of the form*

$$H = U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$$

for pairwise disjoint subsets $\bar{\sigma}_1, \ldots, \bar{\sigma}_p$ of $[n]$. Then

$$[\hat{0}, H] \cong [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

In particular,

$$|\Delta(\hat{0}, H)| \simeq |\Delta(\hat{0}, U_{\bar{\sigma}_1})| * \cdots * |\Delta(\hat{0}, U_{\bar{\sigma}_p})| * \mathbb{S}^{p-2},$$

where "$*$" denotes join of spaces.

Proof. We will use induction on p. If $K \leq H$ in L_Δ, then K has the form $K' \cap K''$ where $K' \leq U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_{p-1}}$ and $K'' \leq U_{\bar{\sigma}_p}$. Thus $[\hat{0}, H]$ is isomorphic to $[\hat{0}, U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_{p-1}}] \times [\hat{0}, U_{\bar{\sigma}_p}]$ as posets via the map $K' \cap K'' \mapsto (K', K'')$. By the induction hypothesis, the first result is obtained. The second result is obtained from [37, Theorem 5.1]. □

The next lemma, whose proof is completely straightforward and omitted, shows that the lower interval $[\hat{0}, U_{\bar{\sigma}}]$ is isomorphic to the intersection lattice for the diagonal arrangement corresponding to $\mathrm{link}_\Delta \sigma$.

Lemma 3.4.5. *Let $U_{\bar{\sigma}}$ be a subspace in L_Δ for some face σ of Δ. Then the lower interval $[\hat{0}, U_{\bar{\sigma}}]$ is isomorphic to the intersection lattice of the diagonal arrangement $\mathcal{A}_{\mathrm{link}_\Delta(\sigma)}$ corresponding to $\mathrm{link}_\Delta(\sigma)$ on the vertex set $\bar{\sigma}$.*

The following lemma shows that upper intervals in L_Δ are at least still *homotopy equivalent* to the intersection lattice of a diagonal arrangement.

Lemma 3.4.6. *Let $U_{\bar{\sigma}}$ be a subspace in L_Δ for some face σ of Δ. Then the upper interval $[U_{\bar{\sigma}}, \hat{1}]$ is homotopy equivalent to the intersection lattice of the diagonal arrangement $\mathcal{A}_{\Delta_\sigma}$ corresponding to the simplicial complex Δ_σ on the vertex set $\sigma \cup \{v\}$ whose facets are obtained in the following way:*

(A) If $F \cap \sigma$ is maximal among

$$\{F \cap \sigma \mid F \textit{ is a facet of } \Delta \textit{ with } \sigma \nsubseteq F \textit{ and } F \cup \sigma \neq [n]\},$$

then $\widetilde{F} = F \cap \sigma$ is a facet of Δ_σ.

(B) If a facet F of Δ satisfies $F \cup \sigma = [n]$, then $\widetilde{F} = (F \cap \sigma) \cup \{v\}$ is a facet of Δ_σ.

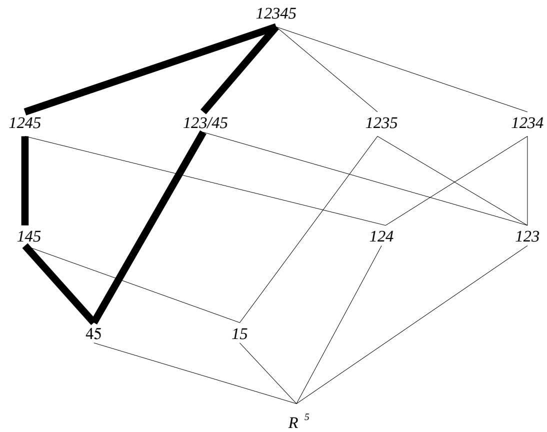
Proof. We apply a standard crosscut/closure lemma ([7, Theorem 10.8]) saying that a finite lattice L is homotopy equivalent to the sublattice consisting of the joins of subsets of its atoms. By the closure relation on $[U_{\bar{\sigma}}, \hat{1}]$ which sends a subspace to the intersection of all subspaces that lie weakly below it and cover $U_{\bar{\sigma}}$ in $[U_{\bar{\sigma}}, \hat{1}]$, one can see that $[U_{\bar{\sigma}}, \hat{1}]$ is homotopy equivalent to the sublattice L_σ generated by the subspaces of $[U_{\bar{\sigma}}, \hat{1}]$ that cover $U_{\bar{\sigma}}$. By using the map ψ defined by

$$\psi(U_{\bar{\tau}}) = \begin{cases} U_{(\bar{\tau}-\bar{\sigma})\cup\{v\}}, & \text{if } \bar{\sigma} \cap \bar{\tau} \neq \emptyset, \\ U_{\bar{\tau}}, & \text{otherwise,} \end{cases}$$

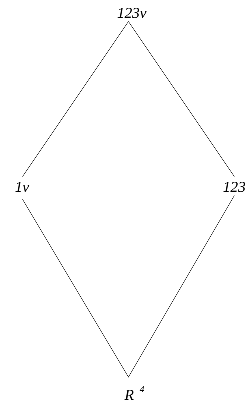
one can see that L_σ is isomorphic to the intersection lattice L_{Δ_σ} for a simplicial complex Δ_σ on the vertex set $\sigma \cup \{v\}$. The facets of Δ_σ correspond the subspaces that cover $U_{\bar{\sigma}}$, giving the claimed characterization of facets of Δ_σ. □

Example 3.4.7. Let Δ be a simplicial complex in Figure 2.1(c), i.e., a simplicial complex on $\{1, 2, 3, 4, 5\}$ with facets 123, 234, 35, 45 and let $\sigma = 123$. Then Δ_σ is a simplicial complex on $\{1, 2, 3, v\}$ and its facets are 23 and v. The intersection lattices L_Δ and L_{Δ_σ} are shown in Figure 3.4 and it is easy to see that the order complex for $\overline{L}_{\Delta_\sigma}$ is homotopy equivalent to the order complex for the interval $(U_{45}, \hat{1})$ in L_Δ. Note that the thick lines in Figure 3.4(a) represent the closed interval $[U_{45}, \hat{1}]$ in L_Δ.

In general, the simplicial complex Δ_σ of Lemma 3.4.6 is not shellable, even though Δ is shellable (see Example 3.1.2). However, the next lemma shows that Δ_σ is shellable if σ is the *last facet* in the shelling order.



(a) L_Δ



(b) L_{Δ_σ}

Figure 3.4: The intersection lattices for Δ and Δ_σ

Lemma 3.4.8. *Let Δ be a shellable simplicial complex. If F is the last facet in a shelling order of Δ, then Δ_F is also shellable. Moreover, if $\widetilde{F_i}$ is a facet of Δ_F of type (B) (defined in Lemma 3.4.6), then $\widetilde{G_i} = G_i \cap F$.*

Proof. One can check that the following gives a shelling order on the facets of Δ_F. First list the facets of type (A) according to the order of their earliest corresponding facet of Δ, followed by the facets of type (B) according to the order of their corresponding facet of Δ.

To see the second assertion, let $\widetilde{F_i}$ be a facet of Δ_F of type (B), i.e.,

$$\widetilde{F_i} = (F_i \cap F) \cup \{v\}$$

for some facet F_i of Δ such that $F \cup F_i = [n]$. Then $\widetilde{G_i} = G_i \cap F$ follows from the observations that $F_i \cap F$ is an old wall of $\widetilde{F_i}$, and all other old walls of $\widetilde{F_i}$ are $\widetilde{F_k} \cap \widetilde{F_i}$ for some facets $\widetilde{F_k}$ of Δ_F of type (B) such that $F_k \cap F_i$ is an old wall of F_i. □

Example 3.4.9. Let Δ be a shellable simplicial complex on $\{1, 2, \ldots, 7\}$ with a shelling 12367, 12346, 13467, 34567, 13457, 14567, 12345 and let $F = 12345$. Then Δ_F is a simplicial complex on $\{1, 2, 3, 4, 5, v\}$ and its facets are $123v$, 1234, $134v, 345v, 1345, 145v$. Since 1234, 1345 are facets of Δ_F of type (A) and $123v$,

$134v$, $345v$, $145v$ are facets of Δ_F of type (B), the ordering 1234, 1345, $123v$, $134v$, $345v$, $145v$ is a shelling of Δ_F.

We next construct the saturated chains appearing in the statement of Theorem 3.4.1.

3.4.1 Constructing the chains $C_{D,w}$

Let Δ be shellable and let $U_{\bar{\sigma}}$ be a subspace in L_Δ. Let

$$D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$$

be a shelling-trapped decomposition of $\bar{\sigma}$ and let w be a permutation on $[p-1]$. We construct a chain $C_{D,w}$ in $[\hat{0}, U_{\bar{\sigma}}]$ as follows:

(i) By Lemma 3.4.2, the interval $[\hat{0}, U_{\bar{\sigma}}]$ contains $U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}$ and the interval $[U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}, U_{\bar{\sigma}}]$ is isomorphic to the set partition lattice Π_p. It is well known that the order complex of $\overline{\Pi}_p = \Pi_p - \{\hat{0}, \hat{1}\}$ is homotopy equivalent to a wedge of $(p-1)!$ spheres of dimension $p-3$ and there is a saturated chain C_w in Π_p for each permutation w of $[p-1]$ such that removing $\{\overline{C}_w = C_w - \{\hat{0}, \hat{1}\} | w \in \mathfrak{S}_{p-1}\}$ from the order complex of $\overline{\Pi}_p$ gives a contractible subcomplex (see [5, Example 2.9]). Identify $U_{\bar{\sigma}_1}, \cdots, U_{\bar{\sigma}_p}$ with $1, \ldots, p$ in this order and take the saturated chain $\widetilde{C}_w$ in $[U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}, U_{\bar{\sigma}}]$ which corresponds to the chain C_w in Π_p.

(ii) By Lemma 3.4.4,

$$[\hat{0}, U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}] \cong [\hat{0}, U_{\bar{\sigma}_1}] \times \cdots \times [\hat{0}, U_{\bar{\sigma}_p}].$$

Since Δ is shellable and $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$ for all j, one can see that $[\hat{0}, U_{\bar{\sigma}_j}]$ has a subinterval $[U_{\overline{F}_{i_j}}, U_{\bar{\sigma}_j}]$ which is isomorphic to the boolean algebra of the set of order $|\bar{\sigma}_j| - |\overline{F}_{i_j}|$. Thus

$$[U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}, U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}]$$

is isomorphic to

$$[U_{\overline{F}_{i_1}}, U_{\bar{\sigma}_1}] \times \cdots \times [U_{\overline{F}_{i_p}}, U_{\bar{\sigma}_p}]$$

and hence is isomorphic to the boolean algebra of the set of order

$$\sum_{j=1}^{p} \left(|\bar{\sigma}_j| - |\overline{F}_{i_j}| \right).$$

Take any saturated chain $\widetilde{C}$ in

$$[U_{\overline{F}_{i_1}/\cdots/\overline{F}_{i_p}}, U_{\bar{\sigma}_1/\cdots/\bar{\sigma}_p}].$$

(iii) Define a saturated chain $C_{D,w}$ by

$$\hat{0} \prec U_{\overline{F}_{i_p}} \prec U_{\overline{F}_{i_p}/\overline{F}_{i_{p-1}}} \prec \cdots \prec U_{\overline{F}_{i_p}/\cdots/\overline{F}_{i_1}}$$

followed by the chains $\widetilde{C}$ and $\widetilde{C}_w$ (where $\prec$ means the covering relation in L_Δ).

Note that the length of the chain $\overline{C}_{D,w} = C_{D,w} - \{\hat{0}, U_{\bar{\sigma}}\}$ is

$$l(\overline{C}_{D,w}) = p(2-n) + \sum_{j=1}^{p} |F_{i_j}| + |\bar{\sigma}| - 3.$$

Example 3.4.10. Let Δ be the shellable simplicial complex in Example 3.4.9. Then one can see that

$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$

is a shelling-trapped decomposition of $\{1, 2, 3, 4, 5, 6, 7\}$. Let w be the permutation in $\mathfrak{S}_2$ with $w(1) = 2$ and $w(2) = 1$. Then the maximal chain C_w in Π_3 corresponding to w is $(1 \mid 2 \mid 3) - (1 \mid 23) - (123)$. By identifying U_{45}, U_{123}, U_{67} with $1, 2, 3$ in this order, one can get

$$\widetilde{C}_w = U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}.$$

Since $[U_{45/23/67}, U_{45/123/67}]$ is isomorphic to a boolean algebra of the set of order 1, one can take

$$\widetilde{C} = U_{45/23/67} \prec U_{45/123/67}.$$

Thus $C_{D,w}$ is the chain

$$\hat{0} \prec U_{67} \prec U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}.$$

The upper interval $(U_{67}, \hat{1})$ is shown in Figure 3.5 and the chain $\overline{C}_{D,w}$ is represented by thick lines.

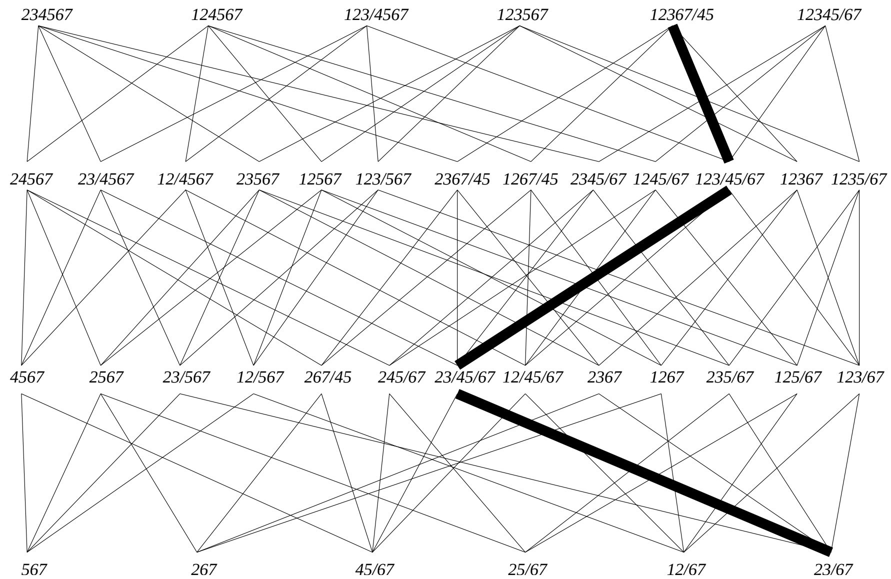


Figure 3.5: The upper interval $(U_{67}, \hat{1})$ in L_Δ

The following lemma provides the relationship between the shelling-trapped decompositions of $[n]$ containing F and the shelling-trapped decompositions of $F \cup \{v\}$.

Lemma 3.4.11. *Let Δ be a shellable simplicial complex and let F be the last facet in the shelling order of Δ. Then there is a one-to-one correspondence between*

- *pairs (D, w) of shelling-trapped decompositions D of $[n]$ over Δ containing F and $w \in \mathfrak{S}_{|D|-1}$, and*
- *pairs $(\widetilde{D}, \tilde{w})$ of shelling-trapped decompositions $\widetilde{D}$ of $F \cup \{v\}$ over Δ_F and $\tilde{w} \in \mathfrak{S}_{|\widetilde{D}|-1}$.*

Moreover, one can choose $\overline{C}_{D,w}$ and $\overline{C}_{\widetilde{D},\tilde{w}}$ so that $\overline{C}_{D,w} - U_{\overline{F}}$ corresponds to $\overline{C}_{\widetilde{D},\tilde{w}}$ under the homotopy equivalence in Lemma 3.4.6.

Proof. Let $D = \{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ be a shelling-trapped decomposition of $[n]$ over Δ with $F_{i_p} = F$ and let w be a permutation in $\mathfrak{S}_{p-1}$. Then one can see that $\widetilde{F}_{i_j} = (F_{i_j} \cap F) \cup \{v\}$ are facets of Δ_F of type (B) for $j = 1, \ldots, p-1$ and $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_{p-1}}$. By Lemma 3.4.8, $\widetilde{G}_{i_j} = G_{i_j} \cap F$ for all $j = 1, \ldots, p-1$.

There are two cases to consider:

Case 1. $\sigma_p \neq F$.

In this case, we will show

$$\widetilde{D} = \{([\bar{\sigma}_p \cap F] \cup \{v\}, \widetilde{F}), (\bar{\sigma}_1, \widetilde{F}_{i_1}), \ldots, (\bar{\sigma}_{p-1}, \widetilde{F}_{i_{p-1}})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$ over Δ_F ($\widetilde{F}$ will be defined later).

Define

$$\tilde{\sigma}_j = \begin{cases} (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, p-1, \\ \sigma_j & \text{for } j = p. \end{cases}$$

For $j = 1, \ldots, p-1$, $\widetilde{G}_{i_j} \subseteq \tilde{\sigma}_j \subseteq \widetilde{F}_{i_j}$ since $G_{i_j} \subseteq \sigma_j \subseteq F_{i_j}$.

Since $G_{i_p} \subseteq \sigma_p$, it must be that σ_p is an intersection of some old walls of F. Thus one can find a family $\mathcal{G}$ of facets of Δ such that $\sigma_p = \cap_{F' \in \mathcal{G}}(F' \cap F)$ and $F' \cap F \lessdot F$ for all $F' \in \mathcal{G}$. Since $|F' \cup F| = |F'| + 1 < n$, one knows that $F' \cap F$ is a facet of Δ_F of type (A) for all $F' \in \mathcal{G}$. Let $\widetilde{F} = F_k \cap F$ be the last facet in the family $\{F' \cap F | F' \in \mathcal{G}\}$ (pick k as small as possible). Since all facets of Δ_F occurring earlier than $\widetilde{F}$ have the form $F \cap F_i$ such that $F_i < F_k$ and $F_i \cap F \lessdot F$, one can see $\widetilde{G} \subseteq \tilde{\sigma}_p \subseteq \widetilde{F}$.

Thus $\widetilde{D}$ is a shelling-trapped decomposition of $F \cup \{v\}$ over Δ_F. Also one can define $\tilde{w} \in \mathfrak{S}_{p-1}$ by

$$\tilde{w}(j) = \begin{cases} w(j-1) & \text{if } 1 < j \leq p-1, \\ w(p-1) & \text{if } j = 1. \end{cases}$$

Case 2. $\sigma_p = F$.

In this case, we claim that

$$\widetilde{D} = \{(\bar{\sigma}_1, \widetilde{F}_{i_1}), \ldots, (\bar{\sigma}_k \cup \{v\}, \widetilde{F}_{i_k}), \ldots, (\bar{\sigma}_{p-1}, \widetilde{F}_{i_{p-1}})\}$$

is a shelling-trapped decomposition of $F \cup \{v\}$.

Let $k = w(1)$. Define

$$\tilde{\sigma}_j = \begin{cases} \sigma_j \cap F & \text{for } j = k, \\ (\sigma_j \cap F) \cup \{v\} & \text{for } j = 1, \ldots, \hat{k}, \ldots, p-1. \end{cases}$$

Then

$$\widetilde{G}_{i_k} \subseteq \tilde{\sigma}_k = \sigma_k \cap F \subseteq \widetilde{F}_{i_k}$$

and

$$\widetilde{G}_{i_j} \subseteq \tilde{\sigma}_j = (\sigma_j \cap F) \cup \{v\} \subseteq \widetilde{F}_{i_j}$$

for $j = 1, \ldots, \hat{k}, \ldots, p-1$. Thus $\widetilde{D}$ is a shelling-trapped decomposition of $F \cup \{v\}$. Also one can define $\tilde{w} \in \mathfrak{S}_{p-2}$ as follows:

$$\tilde{w}(j) = \begin{cases} w(j+1) & \text{if } w(j+1) < k, \\ w(j+1) - 1 & \text{if } w(j+1) > k. \end{cases}$$

Conversely, let

$$\widetilde{D} = \{([F \cup \{v\}] - \tilde{\sigma}_1, \widetilde{F}_{i_1}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, \widetilde{F}_{i_p})\}$$

be a shelling-trapped decomposition of $F \cup \{v\}$, where $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_p}$ are facets of Δ_F, and let $\tilde{w}$ be a permutation in $\mathfrak{S}_{p-1}$. There is at most one $\widetilde{F}_{i_j}$ that does not contain v because $\widetilde{F}_{i_j} \cup \widetilde{F}_{i_k} = F \cup \{v\}$ for all $j \neq k$. Since $\widetilde{F}_{i_1} < \cdots < \widetilde{F}_{i_p}$ and the facets without v appear earlier than the ones with v, there are two possible cases.

Case 1. $v \notin \widetilde{F}_{i_1}$ and $v \in \widetilde{F}_{i_j}$ for $j = 2, \ldots, p$.

In this case, $v \notin \tilde{\sigma}_p$, i.e., $v \in [F \cup \{v\}] - \tilde{\sigma}_p$. One can show that a family

$$D = \{([F \cup \{v\}] - \tilde{\sigma}_2, F_{i_2}), \ldots, ([F \cup \{v\}] - \tilde{\sigma}_p, F_{i_p}), ([n] - \tilde{\sigma}_1, F)\},$$

where $F_{i_j} = (\widetilde{F}_{i_j} \cup F) - \{v\}$ for $j = 2, \ldots, p$, is a shelling-trapped decomposition of $[n]$ and $w \in \mathfrak{S}_{p-1}$ is defined by

$$w(j) = \begin{cases} \tilde{w}(j+1) & \text{if } 1 \leq j < p-1, \\ \tilde{w}(1) & \text{if } j = p-1. \end{cases}$$

Case 2. $v \in \widetilde{F}_{i_j}$ for $j = 1, \ldots, p$.

In this case, there is a k such that $v \in [F \cup \{v\}] - \tilde{\sigma}_k$. One can show that a family

$$D = \{(F - \tilde{\sigma}_1, F_{i_1}), \ldots, (F - \tilde{\sigma}_p, F_{i_p}), ([n] - F, F)\},$$

where $F_{i_j} = (\widetilde{F}_{i_j} \cup F) - \{v\}$ for $j = 2, \ldots, p$, is a shelling-trapped decomposition of $[n]$ and $w \in \mathfrak{S}_p$ can be defined by

$$w(j) = \begin{cases} \tilde{w}(j-1) & \text{if } 1 < j \text{ and } \tilde{w}(j-1) < k, \\ \tilde{w}(j-1)+1 & \text{if } 1 < j \text{ and } \tilde{w}(j-1) \geq k, \\ k & \text{if } j = 1. \end{cases}$$

Proof of the second assertion is straightforward. $\square$

Example 3.4.12. Let Δ be the shellable simplicial complex in Example 3.4.9. In Example 3.4.10, $C_{D,w}$ is the chain

$$\hat{0} \prec U_{67} \prec U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367} \prec U_{1234567}$$

for the shelling-trapped decomposition

$$D = \{(45, F_1 = 12367), (123, F_6 = 14567), (67, F_7 = 12345)\}$$

of $\{1, 2, 3, 4, 5, 6, 7\}$ and the permutation w in $\mathfrak{S}_2$ with $w(1) = 2$ and $w(2) = 1$.

Since $67 = \overline{F}_7$, the corresponding shelling-trapped decomposition $\widetilde{D}$ of the set $\{1, 2, 3, 4, 5, v\}$ is

$$\widetilde{D} = \{(45, \widetilde{F}_1 = 123v), (123v, \widetilde{F}_6 = 145v)\}$$

and the corresponding permutation $\tilde{w} \in \mathfrak{S}_1$ is the identity.

The map ψ from the proof of Lemma 3.4.6 sends the chain

$$U_{23/67} \prec U_{45/23/67} \prec U_{45/123/67} \prec U_{45/12367}$$

to the chain

$$U_{23} \prec U_{45/23} \prec U_{45/123} \prec U_{45/123v}.$$

and this chain satisfies the conditions for $\overline{C}_{\widetilde{D},\tilde{w}}$.

The intersection lattice for Δ_F is shown in Figure 3.6 and the chain $\overline{C}_{\widetilde{D},\tilde{w}}$ is represented by thick lines.

Proof of Theorem 3.4.1. By Lemma 3.4.5, it is enough to establish the case when $\bar{\sigma} = \cup_{i=1}^{q} \overline{F}_i = [n]$. Since every chain $\overline{C}_{D,w}$ is saturated, it is enough to show

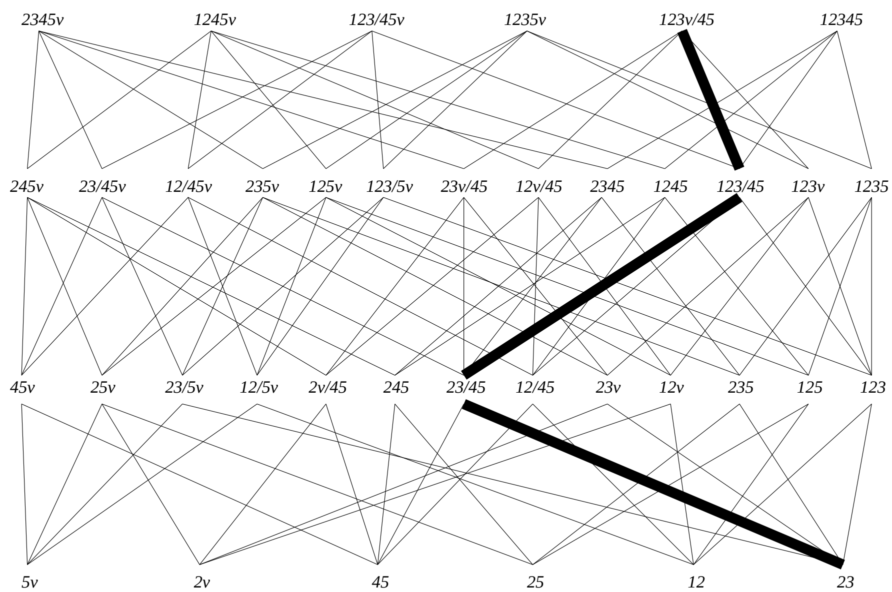


Figure 3.6: The interval $(\hat{0}, \hat{1})$ in L_{Δ_F}

that $\widehat{\Delta}(\overline{L}_\Delta)$, the simplicial complex obtained after removing the corresponding simplices for all pairs (D, w), is contractible. We use induction on the number q of facets of Δ.

Base case: $q = 2$. If Δ has only two facets F_1 and F_2 and $\overline{F}_1 \cup \overline{F}_2 = [n]$, then F_2 has only one element and $G_2 = \emptyset$. It is easy to see that the order complex $\Delta(\overline{L}_\Delta)$ is homotopy equivalent to $\mathbb{S}^0$ and $D = \{([n], F_2)\}$ is the only shelling-trapped decomposition of $[n]$, while $\overline{C}_{D,\emptyset} = (U_{\overline{F}_2})$ is the corresponding saturated chain. Therefore, $\widehat{\Delta}(\overline{L}_\Delta)$ is contractible when $q = 2$.

Inductive step. Now, assume that $\widehat{\Delta}(\overline{L}_\Delta)$ is contractible for all shellable simplicial complexes Δ with less than q facets. For simplicity, denote $L = L_\Delta$. Let $F = F_q$ be the last facet in the shelling order of Δ and $H = U_{\overline{F}}$. Let L' be the intersection lattice for Δ', where Δ' is the simplicial complex generated by the facets $F_1, \ldots, F_{q-1}$.

Let $\overline{L}_{\geq H}$ denote the subposet of elements in $\overline{L}$ which lie weakly above H. Consider the decomposition of $\widehat{\Delta}(\overline{L}) = X \cup Y$, where X is the simplicial complex obtained by removing all simplices corresponding to chains $\overline{C}_{D,w}$ and $\overline{C}_{D,w} - H$ from $\Delta(\overline{L}_{\geq H})$ for all $\overline{C}_{D,w}$ containing H, and Y is the simplicial complex obtained by removing all simplices corresponding to chains $\overline{C}_{D,w}$ not containing H from $\Delta(\overline{L}-\{H\})$. Our goal will be to show that X, Y and $X\cap Y$ are all contractible, and hence so is $X\cup Y (=\widehat{\Delta}(L))$.

Step 1. Contractibility of X

Since X has a cone point H, it is contractible.

Step 2. Contractibility of Y

Define the closure relation π on L which sends a subspace to the join of the elements covering $\hat{0}$ which lie below it except H. Then the closed sets form a sublattice of L, which is the intersection lattice L' for the diagonal arrangement corresponding to Δ'. It is known that the inclusion of closed sets $\overline{L}\cap L' \hookrightarrow \overline{L}-\{H\}$ is a homotopy equivalence (see [8, Lemma 7.6]). We have to consider the following two cases:

Case 1. $S := \cup_{i=1}^{q-1}\overline{F}_i \neq [n]$,

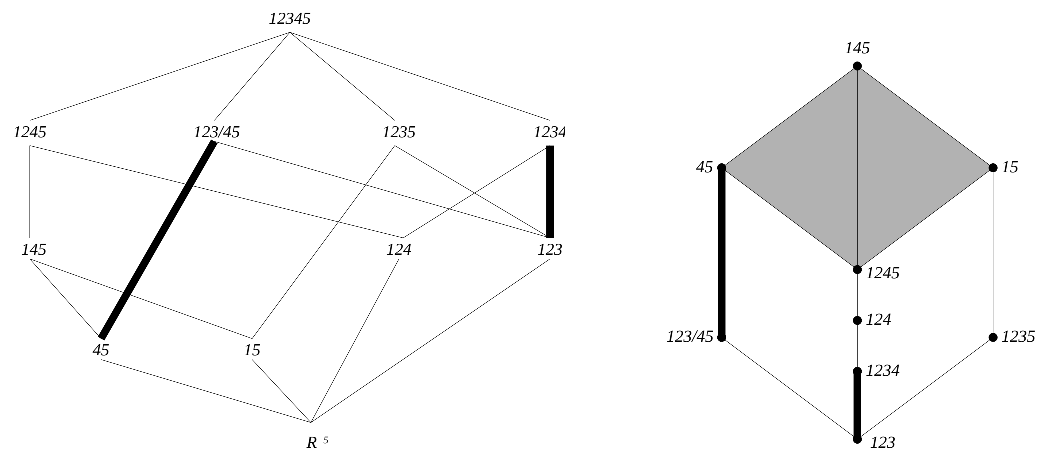
Then $\overline{L}\cap L' = L' - \{\hat{0}\}$ since $U_S \in \overline{L}$. Since $\overline{L}\cap L'$ has a cone point U_S, it is contractible. Since $S \neq [n]$, there is no shelling-trapped decomposition of $[n]$ for Δ'. Thus Y is contractible.

Case 2. $S := \cup_{i=1}^{q-1}\overline{F}_i = [n]$,

In this case, $\overline{L}\cap L' = \overline{L'}$. Moreover, $\widehat{\Delta}(\overline{L'})$ is homotopy equivalent to Y since every element in a chain $\overline{C}_{D,w}$ in $\widehat{\Delta}(\overline{L}-\{H\})$ is fixed under π. Since Δ' has $q-1$ facets, the induction hypothesis implies that $\widehat{\Delta}(\overline{L'})$ is contractible and hence so is Y.

Step 3. Contractibility of $X\cap Y$

Note that $X\cap Y$ is obtained by removing simplices corresponding to $\overline{C}_{D,w}-H$ for all $\overline{C}_{D,w}$ containing H from $\overline{L}_{>H}$. By Lemma 3.4.8, $(\overline{L})_{>H}$ is isomorphic to the proper part of the intersection lattice L_F for the diagonal arrangement corresponding to Δ_F on $F\cup\{v\}$. Also, Lemma 3.4.11 implies that $X\cap Y$ is isomorphic to $\widehat{\Delta}(\overline{L}_F)$, where $\widehat{\Delta}(\overline{L}_F)$ is obtained by removing simplices corresponding to $\overline{C}_{\widetilde{D},\tilde{w}}$ for all shelling-trapped decomposition $\widetilde{D}$ of $F\cup\{v\}$ and $\tilde{w}\in\mathfrak{S}_{|\widetilde{D}|-1}$ from $\overline{L}_F$. Since Δ_F has fewer facets than Δ, the induction hypothesis implies $\widehat{\Delta}(\overline{L}_F)$ is



(a) L_Δ (b) The order complex of $\overline{L}_\Delta$

Figure 3.7: The intersection lattice L_Δ of $\mathcal{A}_\Delta$ and the order complex of $\overline{L}_\Delta$

contractible and hence $X \cap Y$ is also contractible. $\square$

Example 3.4.13. Let Δ be a simplicial complex in Figure 2.1(c). Then $F_1 = 123$, $F_2 = 234$, $F_3 = 35$ and $F_4 = 45$ is a shelling of Δ and

$$G_1 = 123, G_2 = 23, G_3 = 3, \text{ and } G_4 = \emptyset.$$

Let $\bar{\sigma} = 12345$. Then $\{(12345, F_4)\}$ and $\{(45, F_1), (123, F_4)\}$ are two possible (unordered) shelling-trapped decompositions of $\bar{\sigma}$ (see Table 3.2). Thus Theorem 3.4.1 implies $\Delta(\hat{0}, U_{12345})$ is homotopy equivalent to a wedge of two circles. The intersection lattice L_Δ and the order complex for its proper part are shown in Figure 3.7. Note that the chains $C_{D,w}$ and the simplices corresponding to each shelling-trapped decomposition are represented by thick lines.

From Theorem 2.3.2 and our results in this section, one can deduce the following.

Corollary 3.4.14. *Let Δ be a shellable simplicial complex on $[n]$. The singularity link of $\mathcal{A}_\Delta$ has the homotopy type of a wedge of spheres, consisting of $p!$ spheres of dimension*

$$n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2$$

Shelling-trapped Decomp.	dim	Shelling-trapped Decomp.	dim
$\{(45, F_1)\}$	3	$\{(15, F_2)\}$	3
$\{(145, F_2)\}$	3	$\{(124, F_3)\}$	2
$\{(1245, F_3)\}$	2	$\{(123, F_4)\}$	2
$\{(1234, F_4)\}$	2	$\{(1235, F_4)\}$	2
$\{(12345, F_4)\}$	2	$\{(45, F_1), (123, F_4)\}$	2

Table 3.2: Shelling-trapped decompositions for Δ

for each shelling-trapped decomposition $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ *of some subset of* $[n]$.

Proof sketch. This is a straightforward, but tedious, calculation. One needs to understand homotopy types of $\Delta(\hat{0}, H)$ for $H \in L_\Delta$ by Theorem 2.3.2. Lemmas 3.4.2 and 3.4.4 reduce this to the case of $\Delta(\hat{0}, U_{\bar{\sigma}})$, which is described fully by Theorem 3.4.1. The rest is some bookkeeping about shelling-trapped decompositions. □

Example 3.4.15. Let Δ be a simplicial complex in Figure 2.1(c). In Example 3.4.13, we show

$$F_1 = 123, F_2 = 234, F_3 = 35, F_4 = 45$$

is a shelling of Δ and

$$G_1 = 123, G_2 = 23, G_3 = 3, G_4 = \emptyset.$$

Table 3.2 shows shelling-trapped decompositions $\{(\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p})\}$ of subsets of $\{1, 2, 3, 4, 5\}$ with corresponding dimensions

$$n + p(2 - n) + \sum_{j=1}^{p} |F_{i_j}| - 2.$$

Therefore Corollary 3.4.14 shows that the singularity link of $\mathcal{A}_\Delta$ is homotopy equivalent to a wedge of three 3-dimensional spheres and eight 2-dimensional spheres.

3.5 Homology of diagonal arrangements

In this section, we show the homology version of Corollary 3.4.14 without using Theorem 3.1.1. This is what motivated us to prove the stronger Corollary 3.4.14 and eventually Theorem 3.1.1.

Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{F}$. Note that the field $\mathbb{F}$ need not have anything to do with the field $\mathbb{R}$ or $\mathbb{C}$ where the arrangement lives.

Let Γ be a simplicial complex on a vertex set $[n] = \{1, 2, \ldots, n\}$. The *Stanley-Reisner ideal* I_Γ is defined to be the ideal of S generated by the set of monomials

$$\{x_{i_1} x_{i_2} \cdots x_{i_r} : i_1 < i_2 < \cdots < i_r, \{i_1, i_2, \ldots, i_r\} \notin \Gamma\},$$

and the *Stanley-Reisner ring* $\mathbb{F}[\Gamma]$ is defined to be $\mathbb{F}[\Gamma] := S/I_\Gamma$ (see [34]).

Note that I_Γ is a *monomial ideal*, i.e., an ideal generated by monomials and $\mathbb{F}[\Gamma]$ is a vector space over $\mathbb{F}$ with basis

$$\{x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_r}^{e_r} : \{i_1, i_2, \ldots, i_r\} \in \Gamma, e_1, e_2, \ldots, e_r > 0\}.$$

In particular, $\mathbb{F}[\Gamma]$ is a graded ring.

The homology Tor group $\mathrm{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})$ can be computed from a free resolution of $\mathbb{F}$ over S/I by considering $\mathbb{F}$ as a trivial S/I-module $\mathbb{F} = (S/I)/(x_1, \ldots, x_n)$. Since S/I is $\mathbb{N}^n$-graded (for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the corresponding graded piece is the linear span of the monomial $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$), this resolution may also be chosen $\mathbb{N}^n$-graded. For a monomial $\boldsymbol{x}^\alpha$, we denote by $\mathrm{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})_\alpha$ or $\mathrm{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})_{x^\alpha}$ the α-graded piece of $\mathrm{Tor}_n^{S/I}(\mathbb{F}, \mathbb{F})$.

Peeva, Reiner and Welker [30] show the following proposition.

Proposition 3.5.1. *Let Δ be a simplicial complex on $[n] = \{1, 2, \ldots, n\}$. Then*

$$H_i(\mathcal{V}^\circ_{\mathcal{A}_\Delta}; \mathbb{F}) = \mathrm{Tor}_{i-2}^{S/I}(\mathbb{F}, \mathbb{F})_{x_1 \cdots x_n}.$$

There are several ways to evaluate $\mathrm{Tor}_i^{S/I}(\mathbb{F}, \mathbb{F})_\alpha$ in commutative algebra. We give one of them which uses Poincaré series.

The multigraded *Poincaré series* of $\mathbb{F}$ is

$$\mathrm{Poin}_{S/I}^{\mathbb{F}}(t, \boldsymbol{x}) = \sum_{i \geq 0, \alpha \in \mathbb{N}^n} \dim_{\mathbb{F}} \mathrm{Tor}_i^{S/I}(\mathbb{F}, \mathbb{F})_\alpha \, t^i \boldsymbol{x}^\alpha.$$

It was proved by Serre that

$$\mathrm{Poin}^{\mathbb{F}}_{S/I}(t,\boldsymbol{x}) \leq \frac{\prod_{i=1}^{n}(1+tx_i)}{1-tQ_{S/I}(t,\boldsymbol{x})}, \tag{3.1}$$

where the inequality means coefficient-wise comparison of power series and

$$Q_{S/I}(t,\boldsymbol{x}) = \sum_{i\geq 0,\alpha\in\mathbb{N}^n} \dim_{\mathbb{F}} \mathrm{Tor}_i^S(S/I,\mathbb{F})_\alpha \, t^i\boldsymbol{x}^\alpha.$$

A ring S/I is called *Golod* if equality holds in Serre's inequality (3.1). It was shown by Golod that S/I is Golod exactly when certain homology operations (Massey operations) vanish in the Koszul complex computing $\mathrm{Tor}_i^S(S/I,\mathbb{F})$ (see [21]).

It is known that if a homogeneous ideal I has a *linear resolution* as an S-module, then S/I is Golod (see [2]). Also Herzog, Reiner and Welker [22] proved that, if I is a *componentwise linear ideal*, then the ring S/I is Golod. They also observed that, for squarefree monomial ideals $I = I_\Gamma$, componentwise linearity is equivalent to the dual complex Γ^* being *sequentially Cohen-Macaulay over* $\mathbb{F}$, a notion introduced by Stanley [34]. It is known that if Δ is shellable, then it is sequentially Cohen-Macaulay over all fields $\mathbb{F}$.

If Δ is a shellable simplicial complex, then $\mathbb{F}[\Delta^*]$ is Golod and hence

$$\mathrm{Poin}^{\mathbb{F}}_{\mathbb{F}[\Delta^*]}(t,\boldsymbol{x}) = \frac{\prod_{i=1}^{n}(1+tx_i)}{1-tQ_{\mathbb{F}[\Delta^*]}(t,\boldsymbol{x})}.$$

Therefore

$$\dim_{\mathbb{F}} H_i(\mathcal{V}^\circ_{\mathcal{A}_\Delta};\mathbb{F}) = \left[t^{i-2}x_1\cdots x_n\right] \frac{\prod_{i=1}^{n}(1+tx_i)}{1-tQ_{\mathbb{F}[\Delta^*]}(t,\boldsymbol{x})},$$

where the right hand side means the coefficient of $t^{i-2}x_1\cdots x_n$ in the power series expansion of $\dfrac{\prod_{i=1}^{n}(1+tx_i)}{1-tQ_{\mathbb{F}[\Delta^*]}(t,\boldsymbol{x})}$.

One can make this more explicit when Δ is a shellable complex with the shelling order $F_1,\ldots,F_q$.

Lemma 3.5.2. *Let Δ be a simplicial complex and F be a simplex such that $H_i(\Delta) = 0$ for all $i < \dim F$. Suppose $F\cap\Delta$ is pure of codimension 1 in F. Then*

$$\dim_{\mathbb{F}} H_i(\mathrm{link}_{\Delta\cup\{F\}}(\sigma);\mathbb{F}) = \dim_{\mathbb{F}} H_i(\mathrm{link}_{\Delta}(\sigma);\mathbb{F})$$

unless $\sigma \subset F$, $\operatorname{link}_\Delta(\sigma) \cap (F-\sigma) = \partial(F-\sigma)$ *and* $i = \dim(F-\sigma)$. *In that case,*

$$\dim_{\mathbb{F}} H_i(\operatorname{link}_{\Delta\cup\{F\}}(\sigma);\mathbb{F}) = \dim_{\mathbb{F}} H_i(\operatorname{link}_\Delta(\sigma);\mathbb{F}) + 1.$$

Proof. Adding a new simplex F can't change $\operatorname{link}_{\Delta\cup\{F\}}(\sigma)$ from $\operatorname{link}_\Delta(\sigma)$ unless $\sigma \subset F$. Thus let's assume that $\sigma \subset F$. For the simplicity, let $\hat{\Delta} = \operatorname{link}_\Delta(\sigma)$ and $\hat{F} = F - \sigma$. Then $\operatorname{link}_{\Delta\cup\{F\}}(\sigma) = \hat{\Delta} \cup \hat{F}$. Consider the Mayer-Vietoris sequence of $\{\hat{\Delta}, \hat{F}\}$:

$$\cdots \to H_i(\hat{\Delta}\cap\hat{F}) \to H_i(\hat{\Delta}) \oplus H_i(\hat{F}) \to H_i(\hat{\Delta}\cup\hat{F}) \to H_{i-1}(\hat{\Delta}\cap\hat{F}) \to \cdots.$$

Note that $\hat{F} \notin \hat{\Delta}$, otherwise $\hat{F}\cup\sigma = F \in \Delta$. So, $\hat{\Delta}\cap\hat{F} \subset \partial\hat{F}$. If $\hat{\Delta}\cap\hat{F} \subsetneq \partial\hat{F}$, then $\hat{\Delta}\cap\hat{F}$ is acyclic and $H_i(\hat{\Delta}\cap\hat{F}) = 0$ for all i. By the Mayer-Vietoris sequence, $H_i(\hat{\Delta}\cup\hat{F}) = H_i(\hat{\Delta}) \oplus H_i(\hat{F}) = H_i(\hat{\Delta})$. Now assume that $\hat{\Delta}\cap\hat{F} = \partial\hat{F} = \mathbb{S}^a$ where $a = \dim\hat{F} - 1$. If $i < a$, $H_i(\hat{\Delta}\cap\hat{F}) = 0$. Thus $H_i(\hat{\Delta}\cup\hat{F}) = H_i(\hat{\Delta}) \oplus H_i(\hat{F}) = H_i(\hat{\Delta})$ by the Mayer-Vietoris sequence. For $i = a$, consider the Mayer-Vietoris sequence

$$\cdots \to H_{a+1}(\hat{\Delta}\cap\hat{F}) \to H_{a+1}(\hat{\Delta}) \oplus H_{a+1}(\hat{F}) \to H_{a+1}(\hat{\Delta}\cup\hat{F})$$
$$\to H_a(\hat{\Delta}\cap\hat{F}) \to H_a(\hat{\Delta}) \oplus H_a(\hat{F}) \to H_a(\hat{\Delta}\cup\hat{F}) \to \cdots.$$

Since $\hat{\Delta}\cap\hat{F} = \mathbb{S}^a$, $H_{a+1}(\hat{\Delta}\cap\hat{F}) = 0$ and $H_a(\hat{\Delta}\cap\hat{F}) = \mathbb{F}$. Since $\hat{F}$ is a simplex, $H_{a+1}(\hat{F}) = H_a(\hat{F}) = 0$. Since $a = \dim\hat{F} - 1 < \dim F$, one has $H_a(\hat{\Delta}) = 0$ by assumption. Therefore we get a short exact sequence

$$0 \to H_{a+1}(\hat{\Delta}) \to H_{a+1}(\hat{\Delta}\cup\hat{F}) \to \mathbb{F} \to 0.$$

Hence we have

$$\dim_{\mathbb{F}} H_{a+1}(\hat{\Delta}\cup\hat{F};\mathbb{F}) = \dim_{\mathbb{F}} H_{a+1}(\hat{\Delta};\mathbb{F}) + 1.$$

$\square$

Eagon and Reiner [14] show

$$Q_{\mathbb{F}[\Delta^*]}(t,\boldsymbol{x}) = \sum_{i\geq 0,\sigma\in\Delta} \dim_{\mathbb{F}} \widetilde{H}_{i-2}(\operatorname{link}_\Delta(\sigma);\mathbb{F}) t^i x^{\bar{\sigma}}. \tag{3.2}$$

This result together with Lemma 3.5.2 gives the following Theorem, which is the homology version of Corollary 3.4.14.

Theorem 3.5.3. *Let Δ be a shellable simplicial complex and $F_1, \ldots, F_q$ be a shelling of Δ. Then $\dim_{\mathbb{F}} H_i(\mathcal{V}^{\circ}_{\mathcal{A}_\Delta}; \mathbb{F})$ is the number of ordered shelling-trapped decompositions $((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))$ of some subset of $[n]$ satisfying*

$$i = n + p(2-n) + \sum_{j=1}^{p} |F_{i_j}| - 2.$$

Proof. Let Δ_i is a shellable complex with shelling $F_1, \ldots, F_i$. By the rearrangement lemma [8, Lemma 2.7], we may assume $\dim F_1 \geq \dim F_2 \geq \cdots \geq \dim F_i$. Since $\mathrm{link}_{\Delta_{i-1}}(\sigma) \cap (F_i - \sigma) = \partial(F_i - \sigma)$ if and only if $G_i \subset \sigma$ for $i = 2, \ldots, q$, Lemma 3.5.2 and Equation (3.2) give

$$\begin{aligned} Q_{\mathbb{F}[\Delta^*]}(t,x) &= \sum_{i=1}^{q} \sum_{G_i \subset \sigma \subset F_i} t^{\dim_{\mathbb{F}}(\mathrm{link}_{\Delta_i}(\sigma))+2} x^{\bar{\sigma}} \\ &= \sum_{i=1}^{q} \sum_{G_i \subset \sigma \subset F_i} t^{|F_i|-|\sigma|+2} x^{\bar{\sigma}} \\ &= \sum_{\sigma \in \Delta} \sum_{j \in J_\sigma} t^{|F_j|-n+|\bar{\sigma}|+2} x^{\bar{\sigma}}, \end{aligned}$$

where $J_\sigma = \{j : G_j \subset \sigma \subset F_j\}$. Thus

$$\begin{aligned} & [x_1 \cdots x_n] \, \frac{\prod_{i=1}^{n}(1+tx_i)}{1 - tQ_{\mathbb{F}[\Delta^*]}(t,x)} \\ = & \sum_{S \subset [n]} t^{n-|S|} \left[x^S\right] \left(\sum_{p \geq 0} \left(\sum_{\sigma \in \Delta} \sum_{j \in J_\sigma} t^{|F_j|-n+|\bar{\sigma}|+2} x^{\bar{\sigma}} \right)^p \right) \\ = & \sum_{S \subset [n]} t^{n-|S|} \sum_{((\bar{\sigma}_1, F_{i_1}), \ldots, (\bar{\sigma}_p, F_{i_p}))} \left(t^{|S| + \sum_{j=1}^{p}(|F_{i_j}|-n+2)} \right) \end{aligned}$$

where the last sum is over all ordered shelling-trapped decompositions of S. After the change of the order of sums and suitable degree shift, we get the desired property. $\square$

3.6 $K(\pi, 1)$ examples from matroids

In this section, we give some examples of diagonal arrangements $\mathcal{A}$ where the complement $\mathcal{M}_{\mathcal{A}}$ is $K(\pi, 1)$, coming from rank 3 matroids.

One should note that an arrangement having any subspace of real codimension 1 (hyperplane) will have $\mathcal{M}_\mathcal{A}$ disconnected. So one may assume without loss of generality that all subspaces have real codimension at least 2. Furthermore, if any maximal subspace U in $\mathcal{A}$ has codimension at least 3, then it is not hard to see that $\mathcal{M}_\mathcal{A}$ is not $K(\pi, 1)$. Hence we may assume without loss of generality that all maximal subspaces have real codimension 2.

A hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ is *simplicial* if every chamber in $M_\mathcal{H}$ is a simplicial cone. Davis, Januszkiewicz and Scott [12] show the following theorem.

Theorem 3.6.1. *Let $\mathcal{H}$ be a simplicial real hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{A}$ be any arrangement of codimension-2 subspaces in $\mathcal{H}$ which intersects every chamber in a codimension-2 subcomplex. Then $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$.*

Remark 3.6.2. In order to apply this to diagonal arrangements, we need to consider hyperplane arrangements $\mathcal{H}$ which are subarrangements of the braid arrangement $\mathcal{B}_n$ and also simplicial. It turns out (and we omit the straightforward proof) that all such arrangements $\mathcal{H}$ are direct sums of smaller braid arrangements. So we only consider $\mathcal{H} = \mathcal{B}_n$ itself here.

Corollary 3.6.3. *Let $\mathcal{A}$ be a subarrangement of the 3-equal arrangement of $\mathbb{R}^n$ so that*

$$\mathcal{A} = \{U_{ijk} \mid \{i, j, k\} \in T_\mathcal{A}\},$$

for some collection $T_\mathcal{A}$ of 3-element subsets of $[n]$. Then $\mathcal{A}$ satisfies the hypothesis of Theorem 3.6.1 (and hence $\mathcal{M}_\mathcal{A}$ is $K(\pi, 1)$) if and only if every permutation w in $\mathfrak{S}_n$ has at least one triple in $T_\mathcal{A}$ consecutive.

Proof. It is easy to see that there is a bijection between chambers of $\mathcal{B}_n$ and permutations $w = w_1 \cdots w_n$ in $\mathfrak{S}_n$. Moreover, one can see that each chamber has the form $x_{w_1} > \cdots > x_{w_n}$ with bounding hyperplanes

$$x_{w_1} = x_{w_2}, x_{w_2} = x_{w_3}, \ldots, x_{w_{n-1}} = x_{w_n}$$

and intersects the 3-equal subspaces $x_{w_i} = x_{w_{i+1}} = x_{w_{i+2}}$ for $i = 1, 2, \ldots, n-2$. $\square$

We seek shellable simplicial complexes Δ for which $\mathcal{A}_\Delta$ satisfies this condition.

If Δ is the independent set complex $\mathcal{I}(M)$ for some matroid M, then facets of Δ are bases of M. Simplicial complexes of this kind are called *matroid complexes*, and they are known to be shellable [6]. For a rank 3 matroid M, consider

$$\begin{aligned}\mathcal{A}_{\mathcal{I}(M^\perp)} &= \{U_{ijk} \mid \{i,j,k\} = [n] - B \text{ for some } B \in \mathcal{B}(M^\perp)\} \\ &= \{U_{ijk} \mid \{i,j,k\} \in \mathcal{B}(M)\},\end{aligned}$$

where $M^\perp$ is the dual matroid of M. Note that adding a loop to M does not change the structure of the intersection lattice for $\mathcal{A}_{\mathcal{I}(M^\perp)}$. Thus, if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 3.6.3, then the diagonal arrangement corresponding to the matroid in which all loops have been deleted has $K(\pi,1)$ complement, and hence $\mathcal{A}_{\mathcal{I}(M^\perp)}$ has $K(\pi,1)$ complement.

Definition 3.6.4. Let M be a rank 3 matroid on $[n]$. Say M is *DJS* if $\mathcal{B}(M)$ on the set of all non-loop elements satisfies the condition of Corollary 3.6.3.

The following example shows that matroid complexes are not DJS in general. Thus we look for some subclasses of matroid complexes which are DJS, and hence whose corresponding diagonal arrangements have $K(\pi,1)$ complements; for these, Theorem 3.5.3 gives us the group cohomology $H^\bullet(\pi,\mathbb{Z})$.

Example 3.6.5. Let M be a matroid on $\{1,2,3,4,5,6\}$ which has three distinct parallel classes $\{1,6\}$, $\{2,4\}$ and $\{3,5\}$. Then M is self-dual and $\mathcal{I}(M^\perp)$ is a simplicial complex on $[6]$ whose facets are $123, 134, 145, 125, 236, 256, 346$ and 456. But $w = 124356$ is a permutation that does not satisfy the condition of Corollary 3.6.3.

Recall that a matroid is *simple* if it has no loops nor parallel elements. The following proposition shows that rank 3 simple matroids are DJS.

Proposition 3.6.6. *Let M be a matroid of rank 3 on the ground set $[n]$. If M does not have parallel elements, then M is DJS. In particular, rank 3 simple matroids are DJS.*

Proof. Without loss of generality, we may assume that M is simple. M is not DJS if and only if there is a permutation $w \in \mathfrak{S}_n$ such that every consecutive triple is not in $\mathcal{B}(M)$. Since M is simple, the latter statement is true if and only if each consecutive triple in w form a circuit, i.e., all elements lie on a rank 2 flat. But this is impossible since M has rank 3. $\square$

The following two propositions give some subclasses of matroids with parallel elements which are DJS.

Proposition 3.6.7. *Let M be a rank 3 matroid on the ground set $[n]$ with no circuits of size 3. Let $P_1, \ldots, P_k$ be the distinct parallel classes which have more than one element, and let N be the set of all non-loop elements which are not parallel with anything else. Then, M is DJS if and only if*

$$\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k < |N| - 2.$$

Proof. We may assume that M does not have loops. Since M does not have loops nor circuits of size 3, M is not DJS if and only if one can construct a permutation $w \in \mathfrak{S}_n$ such that for each consecutive triple in w there are at least two elements which are parallel. This means if $w_i \in N$, then $w_{i-2}, w_{i-1}, w_{i+1}$ and w_{i+2} (if they exist) must be in the same parallel class. Such a w can be constructed if and only if $\lfloor \frac{|P_1|}{2} \rfloor + \cdots + \lfloor \frac{|P_k|}{2} \rfloor - k \geq |N| - 2$. $\square$

A simplicial complex Δ on $[n]$ is *shifted* if, for any face σ of Δ, replacing any vertex $i \in \sigma$ by a vertex $j < i$ with $j \notin \sigma$ gives another face in Δ. A matroid M is *shifted* if its independent set complex is shifted. Klivans [24] shows that a rank 3 shifted matroid on the ground set $[n]$ is indexed by some set $\{a, b, c\}$ with $1 \leq a < b < c \leq n$ as follows:

$$\mathcal{B}(M) = \{(a', b', c') : 1 \leq a' < b' < c' \leq n, a' \leq a, b' \leq b, c' \leq c\}.$$

It is not hard to check the following.

Proposition 3.6.8. *Let M be the shifted rank 3 matroid on the ground set $[n]$ indexed by $\{a, b, c\}$. Then, M is DJS if and only if $\lfloor \frac{c-b}{2} \rfloor < a$.*

We have not yet been able to characterize all rank 3 matroids which are DJS.

Chapter 4

Flag enumerations of matroid base polytopes

4.1 Main results

For a matroid M on $[n]$, a *matroid base polytope* $Q(M)$ is the polytope in $\mathbb{R}^n$ whose vertices are the incidence vectors of the bases of M. The polytope $Q(M)$ is a face of a *matroid polytope* first studied by Edmonds [15], whose vertices are the incidence vectors of *all* independent sets in M. In this chapter, we study flags of faces in matroid base polytopes, and their **cd**-index.

It is known that a face σ of a matroid base polytope is the matroid base polytope $Q(M_\sigma)$ for some matroid M_σ on $[n]$ (see [18] and Section 4.2 below). We show that M_σ can be described using equivalence classes of factor-connected flags of subsets of $[n]$. As a result, one can associate a poset for each face of $Q(M)$:

Theorem 4.1.1. *Let M be a matroid on a ground set $[n]$. For a face σ of the matroid base polytope $Q(M)$, one can associate a poset P_σ defined as follows:*

(i) the elements of P_σ are the connected components of M_σ, and

(ii) for distinct connected components C_1 and C_2 of M_σ, $C_1 < C_2$ if and only if

$$C_2 \subset S \subset [n] \text{ and } \sigma \subset H_S \text{ implies } C_1 \subset S,$$

where H_S is the hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = r(S)$.

We find the conditions when a matroid base polytope is split into two matroid base polytopes by a hyperplane:

Theorem 4.1.2. *Let M be a rank r matroid on $[n]$ and H be a hyperplane in $\mathbb{R}^n$ given by $\sum_{e\in S} x_e = k$. Then H decomposes $Q(M)$ into two matroid base polytopes if and only if*

(i) $r(S) \geq k$ and $r(S^c) \geq r - k$,

(ii) if I_1 and I_2 are k-element independent subsets of S such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r-k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$.

The **cd**-index $\Psi(Q)$ of a polytope Q, a polynomial in the noncommutative variables **c** and **d**, is a very compact encoding of the flag numbers of a polytope Q. Generalizing the formula of the **cd**-index of a prism and a pyramid of a polytope given by Ehrenborg and Readdy [16], we give the following theorem:

Theorem 4.1.3. *If Q is a polytope and H is a hyperplane in $\mathbb{R}^n$, then the* **cd***-index of Q satisfies*

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\widehat{Q}) \cdot \mathbf{c} - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot \mathbf{d} \cdot \Psi(\widehat{Q}/\hat{\sigma}),$$

where the sum is over all proper faces σ of Q intersecting both open halfspaces obtained by H nontrivially (notations will be defined in Section 4.4).

We apply these theorems to the **cd**-index of a matroid base polytope for rank 2 matroids.

In Section 4.2, we describe faces of matroid base polytopes using data of matroids. In Section 4.3, we define hyperplane splits of matroid base polytopes and give conditions for hyperplane splits. Section 4.4 contains the definition of **cd**-index and how **cd**-index is changed when a polytope is cut by a hyperplane. We apply these to the **cd**-index of matroid base polytopes for rank 2 matroids in Section 4.5.

4.2 Matroid base polytopes

This section contains the description of faces of matroid base polytopes. In particular, we associate a poset for each face of a matroid base polytope.

We start with a precise characterization of matroid base polytopes. Let $\mathcal{B}$ be a collection of r-element subsets of $[n]$. For each subset $B = \{b_1, \ldots, b_r\}$, let

$$e_B = e_{b_1} + \cdots + e_{b_r} \in \mathbb{R}^n,$$

where e_i is the ith standard basis vector of $\mathbb{R}^n$. The collection $\mathcal{B}$ is represented by the convex hull of these points

$$Q(\mathcal{B}) = \mathrm{conv}\{e_B : B \in \mathcal{B}\}.$$

This is a convex polytope of dimension $\leq n-1$ and is a subset of the $(n-1)$-simplex

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0, x_1 + \cdots + x_n = r\}.$$

Gelfand, Goresky, MacPherson, and Serganova [19] show the following characterization of matroid base polytopes.

Theorem 4.2.1. *$\mathcal{B}$ is the collection of bases of a matroid if and only if every edge of the polytope $Q(\mathcal{B})$ is parallel to an edge of Δ_n.*

For a rank r matroid M on a ground set $[n]$ with a set of bases $\mathcal{B}(M)$, the polytope $Q(M) := Q(\mathcal{B}(M))$ is called the *matroid base polytope* of M.

By the definition, the vertices of $Q(M)$ represent the bases of M. Since every edge of Δ has the form $\mathrm{conv}\{re_\alpha, re_\beta\}$ for some $\alpha \neq \beta$, every edge of $Q(M)$ is parallel to a difference $e_\alpha - e_\beta$ of two standard basis vectors. For two bases B and B' in $\mathcal{B}(M)$, e_B and $e_{B'}$ are connected by an edge if and only if $e_B - e_{B'} = e_\alpha - e_\beta$. Since the latter condition is equivalent to $B - B' = \{\alpha\}$ and $B' - B = \{\beta\}$, the edges of $Q(M)$ represent the basis exchange axiom. The bases exchange axiom gives the following equivalence relation on the ground set $[n]$ of the matroid M: α and β are *equivalent* if there exist bases B and B' in $\mathcal{B}(M)$ with $\alpha \in B$ and $B' = (B - \{\alpha\}) \cup \{\beta\}$. The equivalence classes are called the *connected components* of M. The matroid M is called *connected* if it has only one connected component.

Feichtner and Sturmfels [18] express the dimension of the matroid base polytope $Q(M)$ in terms of the number of connected components of M.

Proposition 4.2.2. *Let M be a matroid on $[n]$. The dimension of the matroid base polytope $Q(M)$ equals $n - c(M)$, where $c(M)$ is the number of connected components of M.*

Theorem 4.2.1 implies that every face of a matroid base polytope is also a matroid base polytope. For a face σ of $Q(M)$, let M_σ denote the matroid on $[n]$ whose matroid base polytope is σ. For $\omega \in \mathbb{R}^n$, let M_ω denote the matroid whose bases $\mathcal{B}(M)$ is the collection of bases of M having minimum ω-weight. Then $Q(M_\omega)$ is the face of $Q(M)$ at which the linear form $\sum_{i=1}^n \omega_i x_i$ attains its minimum.

Let $\mathcal{F}(\omega)$ denote the unique flag of subsets

$$\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]$$

for which ω is constant on each set $S_i - S_{i-1}$ and $\omega|_{S_i - S_{i-1}} < \omega|_{S_{i+1}-S_i}$. The *weight class* of a flag $\mathcal{F}$ is the set of vectors ω such that $\mathcal{F}(\omega) = \mathcal{F}$. Ardila and Klivans [1] show that M_ω depends only on $\mathcal{F}(\omega)$, and hence one can call it $M_{\mathcal{F}}$. They also give the following description of $M_{\mathcal{F}}$.

Proposition 4.2.3. *If $\mathcal{F}$ is a flag of subsets*

$$\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n],$$

then

$$M_{\mathcal{F}} = \bigoplus_{i=1}^{k+1} (M|_{S_i})/S_{i-1}.$$

A flag $\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$ is called *factor-connected* if the matroids $(M|_{S_i})/S_{i-1}$ are connected for all $i = 1, \ldots, k+1$. Proposition 4.2.3 together with Proposition 4.2.2 shows that the dimension of $Q(M_{\mathcal{F}})$ is $n-k-1$ if $\mathcal{F}$ is factor-connected. In particular, facets of $Q(M)$ correspond to factor-connected flags of the form $\emptyset \subset S \subset [n]$. Feichtner and Sturmfels [18] show that there are two types of facets of $Q(M)$:

(i) a facet corresponding to a factor-connected flag $\emptyset \subset F \subset [n]$ for some flat F of M (in this case, the facet is called a *flacet*),

(ii) a facet corresponding to a factor-connected flag $\emptyset \subset S \subset [n]$ for an $(n-1)$-subset S of $[n]$.

Proposition 4.2.4. *Let M be a connected matroid on $[n]$ and*

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

be a factor-connected flag. Then the matroid $(M|_{S_{j+1}})/S_{j-1}$ has at most two connected components for $1 \leq j \leq k$.

(i) If it has one connected component, the flag

$$\mathcal{G} = \{\emptyset =: S_0 \subset \cdots \subset S_{j-1} \subset S_{j+1} \subset \cdots \subset S_{k+1} := [n]\}$$

is factor-connected and $Q(M_{\mathcal{G}})$ covers $Q(M_{\mathcal{F}})$ in the face lattice of $Q(M)$.

(ii) If it has two connected components, then they are $S_{j+1} - S_j$ and $S_j - S_{j-1}$. Moreover, the flag

$$\mathcal{F}' = \{\emptyset =: S_0 \subset \cdots \subset S_{j-1} \subset S'_j \subset S_{j+1} \subset \cdots \subset S_{k+1} := [n]\},$$

where $S'_j = S_{j-1} \cup (S_{j+1} - S_j)$, is factor-connected and $Q(M_{\mathcal{F}'}) = Q(M_{\mathcal{F}})$.

Proof. For $j = 1, \ldots, k$, we have $(M|_{S_{j+1}})/S_j = [(M|_{S_{j+1}})/S_{j-1}]/(S_j - S_{j-1})$ and $(M|_{S_j})/S_{j-1} = [(M|_{S_{j+1}})/S_{j-1}]|_{S_j}$. The first assertion follows from [29, Proposition 4.2.10], and the other assertions are obtained from [29, Proposition 4.2.13] and Proposition 4.2.3. $\square$

Two factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ are said to be *equivalent* if there is a sequence of factor-connected flags $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k = \mathcal{F}'$ such that $\mathcal{F}_i$ is obtained from $\mathcal{F}_{i-1}$ by applying Proposition 4.2.4(ii) for $i = 1, \ldots, k$. We write $\mathcal{F} \sim \mathcal{F}'$ when factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

Lemma 4.2.5. *Let M be a connected matroid on $[n]$ and X, Y be disjoint subsets of $[n]$ such that $X \cup Y = [n]$. Then $\mathcal{B}((M|_X) \oplus (M/X))$ and $\mathcal{B}((M|_Y) \oplus (M/Y))$ are disjoint.*

Proof. Since $X \cap Y = \emptyset$ and $X \cup Y = [n]$, one has $r(X) + r(Y) \geq r([n]) = r(M)$. If $r(X) + r(Y) = r(M)$, then $|B \cap X| = r(X)$ and $|B \cap Y| = r(Y)$ for all $B \in \mathcal{B}(M)$, and hence $M = M|_X \oplus M|_Y$. But then M is not connected. Thus $r(X) + r(Y) > r(M)$. Therefore there is no base which has $r(X)$ elements in X and $r(Y)$ elements in Y, i.e., $\mathcal{B}((M|_X) \oplus (M/X))$ and $\mathcal{B}((M|_Y) \oplus (M/Y))$ are disjoint. □

The following proposition shows that the equivalence classes of factor-connected flags characterize faces of a matroid base polytope.

Proposition 4.2.6. *Let M be a connected matroid on $[n]$. If $\mathcal{F}$ and $\mathcal{F}'$ are two factor-connected flags of subsets of $[n]$ given by*

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\},$$
$$\mathcal{F}' = \{\emptyset =: T_0 \subset T_1 \subset \cdots \subset T_l \subset T_{l+1} := [n]\},$$

then $M_{\mathcal{F}} = M_{\mathcal{F}'}$ if and only if $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

Proof. If $\mathcal{F} \sim \mathcal{F}'$, then $M_{\mathcal{F}} = M_{\mathcal{F}'}$ from Proposition 4.2.4.

For the other direction, suppose that $M_{\mathcal{F}} = M_{\mathcal{F}'}$. Then $\mathcal{F}$ and $\mathcal{F}'$ have the same length since $\dim Q(M_{\mathcal{F}}) = n - k - 1$ and $\dim Q(M_{\mathcal{F}'}) = n - l - 1$.

We will use induction on k. Without loss of generality, we may assume that $S_1 \neq T_1$. We claim that

$$T_1 = S_m - S_{m-1} \text{ for some } m > 1. \tag{4.1}$$

This claim is going to be used both in the base case and the inductive step. Let m be the smallest index such that $T_1 \cap S_m \neq \emptyset$ and let $\alpha \in T_1 \cap S_m$. Suppose that T_1 is not contained in S_m, i.e., there is an element $\beta \in T_1 - S_m$. Since $M|_{T_1}$ is connected, there are $B_1', \widetilde{B}_1' \in \mathcal{B}(M|_{T_1})$ such that $\alpha \in B_1'$ and $\widetilde{B}_1' = B_1' - \{\alpha\} \cup \{\beta\}$. Choose $B_j' \in \mathcal{B}((M|_{T_j})/T_{j-1})$ for $j = 2, \ldots, k+1$. Then

$$B' = B_1' \cup B_2' \cup \cdots \cup B_{k+1}' \in \mathcal{B}(M_{\mathcal{F}'}),$$
$$\widetilde{B}' = \widetilde{B}_1' \cup B_2' \cup \cdots \cup B_{k+1}' \in \mathcal{B}(M_{\mathcal{F}'}).$$

But either B' or $\widetilde{B}'$ is not a base of $M_{\mathcal{F}}$ since

$$|\widetilde{B}' \cap (S_m - S_{m-1})| = |B' \cap (S_m - S_{m-1})| - 1.$$

This contradicts the assumption $M_{\mathcal{F}} = M_{\mathcal{F}'}$. Therefore $T_1 \subset S_m$. Now suppose $(S_m - S_{m-1}) - T_1 \neq \emptyset$, i.e., there is an element $\gamma \in (S_m - S_{m-1}) - T_1$. Since $(M|_{S_m})/S_{m-1}$ is connected, there are bases $B_m, \widetilde{B}_m$ of $(M|_{S_m})/S_{m-1}$ satisfying $\alpha \in B_m$ and $\widetilde{B}_m = B_m - \{\alpha\} \cup \{\gamma\}$. If $B_j \in \mathcal{B}((M|_{S_j})/S_{j-1})$ for all $j \neq m$, then

$$
\begin{aligned}
&B = B_1 \cup \cdots \cup B_{k+1} \in \mathcal{B}(M_{\mathcal{F}}),\\
&\widetilde{B} = B_1 \cup \cdots \cup B_{m-1} \cup \widetilde{B}_m \cup B_{m+1} \cup \cdots \cup B_{k+1} \in \mathcal{B}(M_{\mathcal{F}}).
\end{aligned}
$$

But at least one of them is not in $\mathcal{B}(M_{\mathcal{F}'})$ since $|\widetilde{B} \cap T_1| = |B \cap T_1| - 1$. This is impossible because $M_{\mathcal{F}} = M_{\mathcal{F}'}$. Thus $(S_m - S_{m-1}) - T_1 = \emptyset$, i.e., $T_1 = S_m - S_{m-1}$.

Base case: $k = 1$. Equation (4.1) gives $T_1 = S_2 - S_1$ and $T_1 \cup S_1 = [n]$. Since M has two connected components by Lemma 4.2.5, $\mathcal{F}$ and $\mathcal{F}'$ are equivalent by Proposition 4.2.4(ii).

Inductive step. Now suppose that $k > 1$. Let $\widetilde{\mathcal{F}}$ be a flag

$$\emptyset =: S_0 \subset S_1 \subset S_1 \cup T_1 \subset S_2 \cup T_1 \subset \cdots \subset \underbrace{S_{m-1} \cup T_1}_{=S_m} \subset S_{m+1} \cdots \subset S_{k+1} := [n].$$

We claim that $M|_{S_j \cup T_1} = M|_{S_j} \oplus M|_{T_1}$ for $j = 1, \ldots, m-1$. Let $B_j \in \mathcal{B}(M|_{S_j})$ and $B_1' \in \mathcal{B}(M|_{T_1})$. Since $M_{\mathcal{F}} = \oplus_{i=1}^{k+1}(M|_{S_i})/S_{i-1} = \oplus_{i=1}^{k+1}(M|_{T_i})/T_{i-1} = M_{\mathcal{F}'}$ and $M|_{T_1} = (M|_{S_m})/S_{m-1}$, there is a base $B \in \mathcal{B}(M_{\mathcal{F}})$ such that $B \cap S_j = B_j$ and $B \cap T_1 = B_1'$. Thus $B_j \cup B_1' \in \mathcal{B}(M|_{S_j \cup T_1})$, and hence

$$r(M|_{S_j \cup T_1}) = r(M|_{S_j}) + r(M|_{T_1}).$$

By [27, Lemma 3], we have $M|_{S_j \cup T_1} = M|_{S_j} \oplus M|_{T_1}$. Therefore

$$
\begin{aligned}
(M|_{S_1 \cup T_1})/S_1 &= [M|_{T_1} \oplus M|_{S_1}]/S_1\\
&= (M|_{S_m})/S_{m-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(M|_{S_i \cup T_1})/(S_{i-1} \cup T_1) &= [M|_{T_1} \oplus M|_{S_i}]/(S_{i-1} \cup T_1))\\
&= (M|_{S_i})/S_{i-1}
\end{aligned}
$$

for $i = 2, \ldots, m-1$. Thus $\widetilde{\mathcal{F}}$ is factor-connected and $M_{\widetilde{\mathcal{F}}} = M_{\mathcal{F}}$, i.e., $\widetilde{\mathcal{F}} \sim \mathcal{F}$.

Also, one can show $S_1 = T_l - T_{l-1}$ for some $l > 1$ and the chain $\widetilde{\mathcal{F}}'$ given by

$$\emptyset =: T_0 \subset T_1 \subset S_1 \cup T_1 \subset S_1 \cup T_2 \subset \cdots \subset \underbrace{S_1 \cup T_{l-1}}_{=T_l} \subset T_{l+1} \cdots \subset T_{k+1} := [n]$$

is factor-connected and $M_{\widetilde{\mathcal{F}}'} = M_{\mathcal{F}'}$, i.e., $\widetilde{\mathcal{F}}' \sim \mathcal{F}'$.

By the induction assumption, we have $\widetilde{\mathcal{F}} \sim \widetilde{\mathcal{F}}'$ and hence $\mathcal{F}$ and $\mathcal{F}'$ are equivalent. $\square$

If M is a matroid on $[n]$ and S is a subset of $[n]$, then the hyperplane H_S defined by $\sum_{e \in S} x_e = r(S)$ is a supporting hyperplane of $Q(M)$ and $Q(M) \cap H_S$ is the matroid base polytope for $(M|_S) \oplus (M/S)$. The next lemma tells us when a face of $Q(M)$ is contained in H_S.

Lemma 4.2.7. *Let M be a matroid on $[n]$ and S be a subset of $[n]$. A face σ of $Q(M)$ is contained in H_S if and only if there is a factor-connected flag*

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

such that $S \in \mathcal{F}$ and $\sigma = Q(M_{\mathcal{F}})$.

Proof. If there is a flag $\mathcal{F}$ containing S and $\sigma = Q(M_{\mathcal{F}})$, then every base of $M_{\mathcal{F}}$ is contained in $\mathcal{B}((M|_S) \oplus (M/S))$ and hence σ is contained in H_S.

For the converse, suppose σ is contained in H_S. Since σ is a face of $Q(M)$, there is a factor-connected flag

$$\mathcal{F}' = \{\emptyset =: T_0 \subset T_1 \subset \cdots \subset T_k \subset T_{k+1} := [n]\},$$

such that $\sigma = Q(M_{\mathcal{F}'})$ and $k = n - \dim \sigma - 1$. We claim that $T_i - T_{i-1}$ is contained in either S or $[n] - S$ for $i = 1, \ldots, k+1$. If $(M|_{T_i})/T_{i-1}$ contains a loop, then $T_i - T_{i-1}$ has only one element because $(M|_{T_i})/T_{i-1}$ is connected, hence is contained in either S or $[n] - S$. Now assume $(M|_{T_i})/T_{i-1}$ does not contain a loop. Suppose there are $x \in (T_i - T_{i-1}) \cap S$ and $y \in (T_i - T_{i-1}) - S$. Since $(M|_{T_i})/T_{i-1}$ is connected, there exist bases B_1 and B_2 of $M_{\mathcal{F}'}$ with $x \in B_1$ and $B_2 = (B_1 - \{x\}) \cup \{y\}$. Then $|B_1 \cap S| > |B_2 \cap S|$ and so σ is not contained in H_S, which is a contradiction.

Construct a flag $\mathcal{F}$ as follows: $S_j = S_{j-1} \cup (T_k - T_{k-1})$, where k is the smallest index such that

(i) $T_k - T_{k-1} \subset S - S_j$ if $S - S_j \neq \emptyset$,

(ii) $T_k - T_{k-1} \subset [n] - S_j$ if $S - S_j = \emptyset$.

Then $M_{\mathcal{F}} = M_{\mathcal{F}'}$ and hence $\sigma = Q(M_{\mathcal{F}})$. □

For a face σ of $Q(M)$, let L_σ be the poset of all subsets of $[n]$ which are contained in some factor-connected flag $\mathcal{F}$ with $\sigma = Q(M_{\mathcal{F}})$ ordered by inclusion. The following lemma shows that L_σ is a lattice.

Lemma 4.2.8. *Let M be a matroid on $[n]$ and $S, T \subset [n]$. If $Q(M)$ is contained in H_S and H_T, then it is also contained in $H_{S\cap T}$.*

Proof. Since $Q(M)$ is contained in H_S and H_T, we have

$$r(S \cup T) + r(S \cap T) = r(S) + r(T)$$

(see [29, Lemma 1.3.1]). Suppose that $Q(M)$ is not contained in $H_{S\cap T}$, i.e., there is a base B of M such that $|B \cap (S \cap T)| < r(S \cap T)$. Then

$$\begin{aligned} |B \cap (S \cup T)| &= |B \cap S| + |B \cap T| - |B \cap (S \cap T)| \\ &> r(S) + r(T) - r(S \cap T) = r(S \cup T) \end{aligned}$$

which is impossible. Therefore $Q(M)$ is contained in $H_{S\cap T}$. □

Since L_σ is a sublattice of the Boolean lattice $\mathcal{B}_n$, it is distributive. The fundamental theorem for finite distributive lattices [35] shows that there is a finite poset P for which L_σ is the lattice of order ideals of P.

Definition 4.2.9. For a face σ of $Q(M)$, define a poset P_σ as follows:

(i) The elements of P_σ are the connected components of M_σ, and

(ii) for distinct connected components C_1 and C_2 of M_σ, $C_1 < C_2$ if and only if

$$C_2 \subset S \subset [n] \text{ and } \sigma \subset H_S \text{ implies } C_1 \subset S.$$

Note that P_σ is a well-defined poset. Reflexivity and transitivity are clear. Suppose C_1 and C_2 are distinct components with $C_1 < C_2$. Consider a minimal subset S such that $C_2 \subset S$ and $\sigma \subset H_S$. Then $\sigma \subset H_{S-C_2}$ by Lemma 4.2.7. Since $C_1 \subset S - C_2$ and $C_2 \nsubseteq S - C_2$, we have $C_2 \nless C_1$.

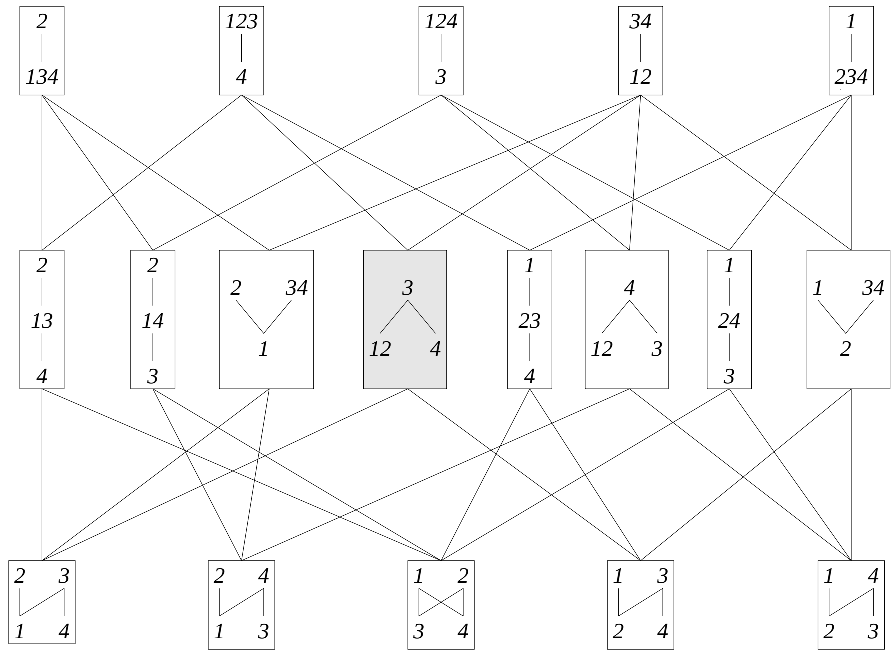


Figure 4.1: The proper part of the face poset of $Q(M_{2,1,1})$

Example 4.2.10. Let $M_{2,1,1}$ be the rank 2 matroid on $[4] = \{1, 2, 3, 4\}$ whose unique non-base is 12 and let σ be an edge of $Q(M_{2,1,1})$ connecting e_{14} and e_{24}. Then the connected components of M_σ are 12, 3 and 4. Since $\{1, 2, 3, 4\}$ is the only subset S containing $\{3\}$ such that $\sigma \subset H_S$, $12 < 3$ and $4 < 3$ in P_σ. One can see that there are no other relations in P_σ. Figure 4.1 is the proper part of the face poset of $Q(M_{2,1,1})$ whose faces are labeled by corresponding posets and P_σ is shown in the shaded box.

Theorem 4.2.11. *Let M be a matroid on $[n]$ and σ be a face of $Q(M)$. Then L_σ is the lattice of order ideals of P_σ.*

Proof. Let S be a set in L_σ. By Lemma 4.2.7, there is a factor-connected flag

$$\mathcal{F} = \{\emptyset =: S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} := [n]\}$$

such that $M_\sigma = M_\mathcal{F}$ and $S_i = S$ for some i. Then $S = \cup_{j=1}^{i}(S_j - S_{j-1})$. Suppose $S_l - S_{l-1} < S_j - S_{j-1}$ in L_σ for some $j \leq i$. Since $\sigma \subset H_S$, $S_l - S_{l-1} \subset S$ and hence S is an order ideal of P_σ.

Conversely, suppose T is an order ideal of P_σ. Let U be the intersection of all subsets $\widetilde{T}$ such that $\sigma \subset H_{\widetilde{T}}$ and $T \subset \widetilde{T}$. By Lemma 4.2.8, U lies in L_σ. Suppose $T \neq U$. By Lemma 4.2.7, there is a factor-connected flag

$$\mathcal{F} = \{\emptyset =: U_0 \subset U_1 \subset \cdots \subset U_k \subset U_{k+1} := [n]\}$$

such that $\sigma = Q(M_\mathcal{F})$ and $U = U_j$ for some j. Since $T \neq U$ and T is an order ideal, we may choose $\mathcal{F}$ so that $U_j - U_{j-1} \nsubseteq T$. Then $U_{j-1} \in L_\sigma$ and $T \subset U_{j-1}$ which is a contradiction. $\square$

Billera, Jia and Reiner [4] define posets related to bases of matroids: For each $e \in B \in \mathcal{B}(M)$ the *basic bond for e in B* is the set of $e' \in [n] - B$ for which $B - \{e\} \cup \{e'\}$ is another base of M. Dually, for each $e' \in [n] - B$ the *basic circuit for e in B* is the set of $e \in B$ for which $B - \{e\} \cup \{e'\}$ is another base of M. Note that e' lies in the basic bond for e if and only if e lies in the basic circuit of e'. Thus one can define the poset P_B to be the poset whose Hasse diagram is the bipartite graph with vertex set $[n]$ and edges directed upward from e to e' whenever e' is in the basic bond for e.

The next proposition shows that our poset P_σ is the same as the poset P_B of Billera, Jia and Reiner if σ is a vertex e_B.

Proposition 4.2.12. *If B is a base of a matroid M on $[n]$ and σ is a vertex e_B of $Q(M)$, then $P_\sigma = P_B$.*

Proof. Since all connected components of M_σ are singletons, P_σ and P_B have the same elements. If $e_1, e_2 \in B$, then they are not comparable in P_σ since $e_B \in H_{e_1}$ and $e_B \in H_{e_2}$. Also, $e_1, e_2 \notin B$ are not comparable in P_σ because $e_B \in H_{B\cup\{e_1\}}$ and $e_B \in H_{B\cup\{e_2\}}$. Now assume $e_1 \in B$ and $e_2 \notin B$. Then $e_2 \not< e_1$ since $e_B \in H_{e_1}$. We claim that $e_1 <_{P_\sigma} e_2$ if and only if $B' := B - \{e_1\} \cup \{e_2\} \in \mathcal{B}(M)$, i.e., $e_1 <_{P_B} e_2$. Assume $e_1 <_{P_\sigma} e_2$, i.e., $e_2 \in S$ and $\sigma_B \in H_S$ implies $e_1 \in S$. Suppose B' is not a base of M. Then $r(B') = r(M) - 1$. Thus we have $e_2 \in B', \sigma_B \in H_{B'}$, but $e_1 \notin B'$, which is a contradiction. Therefore B' is a base of M. Conversely, assume $B' \in \mathcal{B}(M)$. Let S be a subset of $[n]$ such that $e_2 \in S$ and $\sigma_B \in H_S$, i.e., $|B \cap S| = r(S)$. If $e_1 \notin S$, then $|B' \cap S| = |B \cap S| + 1 = r(S) + 1$, which is impossible. Thus $e_1 <_{P_\sigma} e_2$. $\square$

4.3 Matroid base polytope decompositions

In this section, we define the hyperplane split of a matroid base polytope and give conditions when it occurs.

For a matroid M on $[n]$, a *matroid base polytope decomposition* of $Q(M)$ is a decomposition $Q(M) = \cup_{i=1}^{t} Q(M_i)$ where

(i) each M_i is a matroid on $[n]$, and

(ii) for each $i \neq j$, the intersection $Q(M_i) \cap Q(M_j)$ is a proper face of both $Q(M_i)$ and $Q(M_j)$.

A matroid base polytope decomposition of $Q(M)$ is called a *hyperplane split* of $Q(M)$ if $t = 2$. We also say that $Q(M)$ is *decomposable* if there is a matroid base polytope decomposition with $t \geq 2$, and *indecomposable* otherwise.

Consider a hyperplane split $Q(M_1) \cup Q(M_2)$ of a matroid base polytope $Q(M)$. Let $\sum_{i=1}^{n} a_i x_i = b$ be an equation defining the corresponding hyperplane H. Since $Q(M_1) \cap Q(M_2)$ is a matroid base polytope on H and its edges are parallel to $e_i - e_j$ for some $i \neq j$, the only constraints on the normal vector $(a_1, a_2, \ldots, a_n)$ of H are of the form $a_i = a_j$. Using the fact that $Q(M)$ is a subset of a simplex defined by $\sum_{i=1}^{n} x_i = r(M)$ and scaling the right hand side b, one can assume that H is defined by $\sum_{e \in S} x_e = k$ for some subset S of $[n]$.

The following result characterizes when hyperplane splits occur.

Theorem 4.3.1. *Let M be a rank r matroid on $[n]$ and H be a hyperplane defined by $\sum_{e \in S} x_e = k$. Then H gives a hyperplane split of $Q(M)$ if and only if the following are satisfied:*

(i) $r(S) > k$ and $r(S^c) > r - k$,

(ii) if I_1 and I_2 are k-element independent subsets of S such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r - k$, then $(M/I_1)|_{S^c} = (M/I_2)|_{S^c}$.

Remark 4.3.2. Note that if I is a k-element independent subset of S and J is an $(r-k)$-element independent subset of S^c, then I is a base for $(M/J)|_S$ if and only if J is a base for $(M/I)|_{S^c}$. Therefore one can see that the condition (ii) can be replaced with the following condition for S^c:

(ii′) if J_1 and J_2 are $(r-k)$-element independent subsets of S^c such that $(M/J_1)|_S$ and $(M/J_2)|_S$ have rank k, then $(M/J_1)|_S = (M/J_2)|_S$.

Proof of Theorem 4.3.1. Define $\mathcal{B}_k = \{B \in \mathcal{B}(M) : |B \cap S| = k\}$. We will show that the condition (ii) holds if and only if $\mathcal{B}_k$ is a collection of bases of some matroid. Then the assertion follows from Theorem 4.2.1.

Suppose that the condition (ii) is true. Choose any bases B_1 and B_2 in $\mathcal{B}_k$ and $x \in B_1 - B_2$ (without loss of generality, we may assume $x \in B_1 \cap S$). Let $I_i = B_i \cap S$ and $J_i = B_i - S$ for $i = 1, 2$. Then the condition (ii) implies that there is a base $B = I_2 \cup J_1$ in $\mathcal{B}_k$. Since $B_1, B \in \mathcal{B}$, there is $y \in B - B_1 \subset I_2 \subset B_2$ such that $B_3 = B - \{x\} \cup \{y\} \in \mathcal{B}$. Since $y \in I_2 \subset S$, $B_3 \in \mathcal{B}_k$. Thus $\mathcal{B}_k$ forms a collection of bases of a matroid.

Conversely suppose that $\mathcal{B}_k$ is a collection of bases of some matroid. Let I_1 and I_2 be k-element independent subsets of S such that $(M/I_1)|_{S^c}$ and $(M/I_2)|_{S^c}$ have rank $r-k$. Choose $J_1 \in \mathcal{B}((M/I_1)|_{S^c})$ and $J_2 \in \mathcal{B}((M/I_2)|_{S^c})$. Then $B_1 = I_1 \cup J_1$ and $B_2 = I_2 \cup J_2$ are bases for B. We claim that $I_2 \cup J_1$ is also a base of M: this implies $\mathcal{B}((M/I_1)|_{S^c}) \subset \mathcal{B}((M/I_2)|_{S^c})$ and (ii) follows by symmetry. We use induction on the size of $I_1 - I_2$.

Base Case: If $|I_1 - I_2| = 0$, we have $I_2 \cup J_1 = B_1 \in \mathcal{B}$.

Inductive Step: Suppose $|I_1 - I_2| = l$ for some $l \leq k$. Choose an element $x \in I_1 - I_2 \subset B_1 - B_2$. Since $\mathcal{B}_k$ forms a matroid, there exist $y \in I_2 - I_1$ such that $B_3 = B_1 - \{x\} \cup \{y\} \in \mathcal{B}_k \subset \mathcal{B}$. Since $B_3 = (I_1 - \{x\} \cup \{y\}) \cup J_1$, we have $|(B_3 \cap S) - I_2| = l - 1$ and the induction hypothesis implies $I_2 \cup J_1 \in \mathcal{B}$. □

When M has rank 2, Theorem 4.3.1 can be rephrased in the following way.

Corollary 4.3.3. *Let M be a rank 2 matroid on $[n]$ and H be a hyperplane defined by $\sum_{e \in S} x_e = 1$. Then H gives a hyperplane split of $Q(M)$ if and only if S and S^c are both unions of at least two rank one flats.*

4.4 The cd-index

In this section, we define the **cd**-index for Eulerian posets and give the relationship among **cd**-indices of polytopes when a polytope is cut by a hyperplane.

Let P be a graded poset of rank $n+1$ and S be a subset of $[n]$. Define $f_P(S)$ to be the number of chains of P whose ranks are exactly given by the set S. The function $f_P : 2^{[n]} \to \mathbb{N}$ is called the *flag f-vector* of P. The *flag h-vector* is defined by the identity

$$h_P(S) = \sum_{T \subset S} (-1)^{|S-T|} \cdot f_P(T).$$

Since this identity is equivalent to the relation

$$f_P(S) = \sum_{T \subset S} h_P(T),$$

the flag f-vector and the flag h-vector contain the same information.

For a subset S of $[n]$ define the **ab**-monomial $u_S = u_1 \dots u_n$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

The **ab**-*index* of the poset P is defined to be the sum

$$\Psi(P) = \sum_{S \subset [n]} h_P(S) \cdot u_S.$$

An alternative way of defining the **ab**-index is as follows. For a chain

$$c := \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\},$$

we give a *weight* $w_P(c) = w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \dots, \rho(x_k)\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

Define the **ab**-*index* of the poset P to be the sum

$$\Psi(P) = \sum_{c} w(c),$$

where the sum is over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in P. Recall that a poset P is *Eulerian* if its Möbius function μ is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$ (see [35] for more details). One important class of Eulerian posets is face lattices of convex polytopes (see [26, 33]). It is known that the **ab**-index of an Eulerian poset P can be written uniquely as a polynomial of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ (see [3]). When the **ab**-index can be written as a polynomial in **c** and **d**, we call $\Psi(P)$ the **cd**-*index* of P. We will use the notation $\Psi(Q)$ for the **cd**-index of the face poset of a convex polytope Q.

Ehrenborg and Readdy [16] give formulas for the **cd**-index of a pyramid, a prism and a bipyramid of a polytope.

Proposition 4.4.1. *Let Q be a polytope. Then*

$$\Psi(\mathrm{Pyr}(Q)) = \frac{1}{2}\left[\Psi(Q)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(Q) + \sum_{\sigma}\Psi(\sigma)\cdot\mathbf{d}\cdot\Psi(Q/\sigma)\right],$$
$$\Psi(\mathrm{Prism}(Q)) = \Psi(Q)\cdot\mathbf{c} + \sum_{\sigma}\Psi(\sigma)\cdot\mathbf{d}\cdot\Psi(Q/\sigma),$$
$$\Psi(\mathrm{Bipyr}(Q)) = \mathbf{c}\cdot\Psi(Q) + \sum_{\sigma}\Psi(\sigma)\cdot\mathbf{d}\cdot\Psi(Q/\sigma),$$

where the sum is over all proper faces σ of Q.

Note that the **cd**-index of $\mathrm{Bipyr}(Q)$ is obtained from the fact that $\mathrm{Bipyr}(Q)$ is the dual of the prism over the dual of Q and the **cd**-index of the dual polytope is obtained by writing every **ab**-monomial in reverse order.

Let Q be a polytope in $\mathbb{R}^n$. Let H be a hyperplane in $\mathbb{R}^n$ defined by $l(x) = c$ and H^+ (resp. H^-) be the closed halfspace $l(x) \geq c$ (resp. $l(x) \leq c$). Also recall that relint Q is the relative interior of a polytope Q. For simplicity, let $Q^+ := Q \cap H^+$, $Q^- = Q \cap H^-$, and $\widehat{Q} := Q \cap H$. The following theorem provides the relationship among **cd**-indices of polytopes Q, Q^+, Q^- and faces of $\widehat{Q}$.

Theorem 4.4.2. *Let Q be a polytope in $\mathbb{R}^n$ and H be a hyperplane in $\mathbb{R}^n$. Then the following identity holds:*

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\widehat{Q})\cdot\mathbf{c} - \sum_{\sigma}\Psi(\hat{\sigma})\cdot\mathbf{d}\cdot\Psi(\widehat{Q}/\hat{\sigma}),$$

where the sum is over all proper faces σ of Q intersecting both open halfspaces $H^+ - H$ and $H^- - H$ nontrivially.

Proof. Let

$$c = \{\emptyset = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = Q\}$$

be a chain in the face poset of Q. If there is a face σ_i such that relint $\sigma_i \subset H^+ - H$ (resp. relint $\sigma_i \subset H^- - H$), then

$$c^+ = \{\emptyset < \sigma_1 \cap H^+ < \cdots < \sigma_k \cap H^+ < Q^+\}$$

$$(\text{resp. } c^- = \{\emptyset < \sigma_1 \cap H^- < \cdots < \sigma_k \cap H^- < Q^-\})$$

is a corresponding chain in the face poset of Q^+ (resp. Q^-) and $w(c) = w(c^+)$ (resp. $w(c) = w(c^-)$).

Now suppose that relint σ_i is contained in neither $H^+ - H$ nor $H^- - H$ for all $i = 1, \ldots, k$. Let s be the smallest index such that $\sigma_i \not\subseteq H$ and $\hat{c}$ be the chain in the face poset of $\widehat{Q}$ defined by

$$\hat{c} := c \cap H = \{\emptyset = \sigma_0 < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{k+1} = \widehat{Q}\}.$$

(i) Consider a chain c^+ in the face poset of Q^+ such that $c^+ \cap H$ and $\hat{c}$ are the same. Then c^+ is one of the following chains for some j with $s \leq j \leq k+1$:

$$c_1^+ = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \sigma_j^+ < \cdots < \sigma_k^+ < \sigma_{k+1}^+ = Q^+\},$$
$$c_2^+ = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \hat{\sigma}_j < \sigma_j^+ < \cdots < \sigma_k^+ < \sigma_{k+1}^+ = Q^+\}.$$

Let $\sigma := \sigma_j$. Then relint $\sigma \cap (H^+ - H) \neq \emptyset$ and relint $\sigma \cap (H^- - H) \neq \emptyset$. Let $\hat{c}_1 := \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_{j-1} < \hat{\sigma}\}$ and $\hat{c}_2 := \{\hat{\sigma} < \hat{\sigma}_{j+1} < \cdots < \hat{\sigma}_k < \widehat{Q}\}$. There are two cases:

(a) The first case is when $s \leq j \leq k$. Then the sum of the weights of the chains c_1^+ and c_2^+ is given by

$$w(c_1^+) + w(c_2^+) = w_{[\emptyset,\hat{\sigma}]}(\hat{c}_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[\hat{\sigma},\widehat{Q}]}(\hat{c}_2).$$

Note that this case does not occur when $s = k+1$.

(b) The second case is when $j = k+1$. Then the sum of the weights of the chains is

$$w(c_1^+) + w(c_2^+) = w(\hat{c}) \cdot \mathbf{a}.$$

(ii) Consider a chain c^- in the face poset of Q^- such that $c^- \cap H$ and $\hat{c}$ are the same. Then c^- is one of the following chains for some j with $s-1 \leq j \leq k+1$:

$$c_1^- = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_j < \sigma_{j+1}^- < \cdots < \sigma_k^- < \sigma_{k+1}^- = Q^-\},$$
$$c_2^- = \{\emptyset < \hat{\sigma}_1 < \cdots < \hat{\sigma}_j < \sigma_j^- < \sigma_{j+1}^- < \cdots < \sigma_k^- < \sigma_{k+1}^- = Q^-\}.$$

Let $\sigma := \sigma_j$ and define chains $\hat{c}_1$ and $\hat{c}_2$ as in Case (i). If $j \geq s$, then relint $\sigma \cap (H^+ - H) \neq \emptyset$ and relint $\sigma \cap (H^- - H) \neq \emptyset$. There are three cases:

(a) The first case is when $s \leq j \leq k$. Then the sum of the weights of the chains c_1^- and c_2^- is given by

$$w(c_1^-) + w(c_2^-) = w_{[\emptyset,\hat{\sigma}]}(\hat{c}_1) \cdot \mathbf{b} \cdot \mathbf{a} \cdot w_{[\hat{\sigma},\widehat{Q}]}(\hat{c}_2).$$

Note that this case does not occur when $s = k+1$.

(b) The second case is when $j = k+1$. Then the only possible chain is c_2^- and its weight is

$$w(c_2^-) = w(\hat{c}) \cdot \mathbf{b}.$$

(c) The third case is when $j = s-1$. Since $\sigma_{s-1} \subset H$, we have $\hat{\sigma}_{s-1} = \hat{\sigma}_{s-1}^-$. Therefore the only possible chain is c_1^- and its weight is

$$w(c_1^-) = w_{[\emptyset,\hat{\sigma}]}(\hat{c}_1) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot w_{[\hat{\sigma},\widehat{Q}]}(\hat{c}_2) = w(c).$$

Now summing over all such chains in the face posets of Q^+ and Q^-, we obtain

$$\sum_{c^+} w(c^+) + \sum_{c^-} w(c^-) = \sum_{\hat{c}_1,\hat{c}_2} w_{[\emptyset,\hat{\sigma}]}(\hat{c}_1) \cdot \mathbf{d} \cdot w_{[\hat{\sigma},\widehat{Q}]}(\hat{c}_2) + w(\hat{c}) \cdot \mathbf{c} + w(c),$$

where the first (resp. second) sum is over all chains c^+ (resp. c^-) in the face poset of Q^+ (resp. Q^-) such that $c^+ \cap H = \hat{c}$ (resp. $c^- \cap H = \hat{c}$), and the third sum is over all pairs of chains $\hat{c}_1, \hat{c}_2$ obtained above such that $\hat{c}_2$ is nontrivial. Finally, summing over all chains in the face poset of Q, we prove the theorem. □

Remark 4.4.3. The formula for the prism of a polytope in Proposition 4.4.1 is a special case of Theorem 4.4.2, since in this case

$$Q = \mathrm{Prism}(Q') \simeq Q' \times [0, 0.5] = Q^-$$
$$\simeq Q' \times [0.5, 1] = Q^+$$

and $Q' \simeq Q' \times \{0.5\} = \widehat{Q}$.

Also, the formula for the pyramid of a polytope is obtained from Theorem 4.4.2 by considering $Q = \mathrm{Bipyr}(Q')$ split by the hyperplane containing Q': in this case, $Q^+ = Q^- = \mathrm{Pyr}(Q')$ and there are no faces of $\mathrm{Bipyr}(Q')$ intersecting both open halfspaces nontrivially.

Problem 4.4.4. When $Q(M_1) \cup Q(M_2)$ is a hyperplane split of $Q(M)$ with a corresponding hyperplane H, restate Theorem 4.4.2 in terms of matroids.

4.5 Rank 2 matroids

In this section we apply Theorem 4.3.1 and Theorem 4.4.2 to the **cd**-index of a matroid base polytope when a matroid has rank 2.

A (loopless) rank 2 matroid M on $[n]$ is determined up to isomorphism by the composition $\alpha(M)$ of $[n]$ that gives the sizes α_i of its parallelism classes.

Let $\alpha := \alpha_1, \alpha_2, \ldots, \alpha_k$ be a composition of n with the length $l(\alpha) = k$ and let M_α be the corresponding rank 2 matroid on $[n]$. Note that

(i) M_α is decomposable and connected if $k > 3$.

(ii) M_α is indecomposable and connected if $k = 3$.

(iii) M_α is indecomposable with two connected components if $k = 2$. Moreover, we have

$$Q(M_\alpha) \cong Q(U_{1,\alpha_1}) \times Q(U_{1,\alpha_2}) \cong \Delta_{\alpha_1} \times \Delta_{\alpha_2},$$

where $U_{1,n}$ is the uniform matroid of rank 1 on n elements and Δ_n is an $(n-1)$-dimensional simplex.

For two weak compositions (i.e., compositions allowing 0 as parts) α and β of the same length, we define $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $i = 1, 2, \ldots, l(\alpha)$. Let $\bar{\beta}$ be the composition obtained from β by deleting 0 parts. If $\alpha = (2, 4, 0, 6, 7)$ and $\beta = (1, 3, 0, 6, 3)$, then $\beta < \alpha$ and $\bar{\beta} = (1, 3, 6, 3)$.

The following proposition expresses the **cd**-index of a matroid base polytope of a rank 2 matroid M with composition $\alpha(M) = \alpha$ in terms of **cd**-indices of matroid base polytopes of corresponding indecomposable matroids. For simplicity, we use the following notations:

$$\lambda(\alpha,i)=\left(\sum_{j=1}^{i-1}\alpha_j,\alpha_i,\sum_{j=i+1}^{l(\alpha)}\alpha_j\right)\quad \text{for } 2\le i\le l(\alpha)-1,$$
$$\mu(\alpha,i)=\left(\sum_{j=1}^{i}\alpha_j,\sum_{j=i+1}^{l(\alpha)}\alpha_j\right)\qquad \text{for } 1\le i\le l(\alpha)-1.$$

If $\alpha=(2,4,0,6,7)$, then $\lambda(\alpha,4)=(6,6,7)$ and $\mu(\alpha,4)=(12,7)$.

Proposition 4.5.1. *Let α be a composition of n with at least three parts and M_α be the corresponding rank 2 matroid on $[n]$. Then the* **cd***-index of $Q(M_\alpha)$ can be expressed as follows:*

$$\Psi(Q(M_\alpha))=\sum_{i=2}^{l(\alpha)-1}\Psi(Q(M_{\lambda(\alpha,i)}))-\left(\sum_{i=2}^{l(\alpha)-2}\Psi(\Delta_{\mu(\alpha,i)_1}\times\Delta_{\mu(\alpha,i)_2})\right)\cdot\mathbf{c}$$
$$-\sum_{\substack{\beta<\alpha\\ l(\bar\beta)\ge 4}}\prod_{j=1}^{l(\alpha)}\binom{\alpha_j}{\beta_j}\left(\sum_{i=2}^{l(\bar\beta)-2}\Psi(\Delta_{\mu(\bar\beta,i)_1}\times\Delta_{\mu(\bar\beta,i)_2})\right)\cdot\mathbf{d}\cdot\Psi(\Delta_{n-|\bar\beta|}).$$

Proof. After the relabeling, one may assume that M_α has parallelism classes $P_1,\dots,P_k$ such that $\max P_i<\min P_{i+1}$ for all $i=1,\dots,k-1$ and $|P_i|=\alpha_i$ for all $i=1,\dots,k$. Define corresponding indecomposable matroids on $[n]$ as follows:

- M_i^3 is the matroid with three parallelism classes $\cup_{j=1}^{i-1}P_j$, P_i, and $\cup_{j=i+1}^{k}P_j$,
- M_i^2 is the matroid with two parallelism classes $\cup_{j=1}^{i}P_j$ and $\cup_{j=i+1}^{k}P_j$.

Then M_i^3 is isomorphic to $M_{\lambda(\alpha,i)}$ and M_i^2 is isomorphic to $M_{\mu(\alpha,i)}$. For a subset S of $[n]$ such that $M|_S$ is connected, one can also define $(M|_S)_i^3$ and $(M|_S)_i^2$ by considering $M|_S$ as a matroid on $[n]$ such that the elements in $[n]-S$ are loops.

We will use induction on k.

Base Case: If $k=3$, then $Q(M)$ is indecomposable and hence there is nothing to prove.

Inductive Step: From Corollary 4.3.3, one can see that a hyperplane H defined by $\sum_{e\in P_1\cup\cdots\cup P_{k-2}} x_e = 1$ gives a hyperplane split $Q(M^+)\cup Q(M^-)$ of $Q(M)$ where M^+ is a matroid with parallelism classes $P_1, \ldots, P_{k-2}, P_{k-1}\cup P_k$ and M^- has parallelism classes $P_1\cup\cdots\cup P_{k-2}, P_{k-1}, P_k$. Also, $Q(M^+)\cap Q(M^-) = Q(M^0)$ where M^0 is a matroid with parallelism classes $P_1\cup\cdots\cup P_{k-2}$ and $P_{k-1}\cup P_k$. Note that there is no flacet (i.e., facet corresponding to a flat of M) of $Q(M)$ which intersects both open halfspace given by H nontrivially since every flacet of $Q(M)$ corresponds to a base set of the form

$$\{B\in\mathcal{B}(M) : |B\cap P_i| = 1\}$$

for some i. (The corresponding flacet has empty intersection with the open halfspace $H^- - H$ if $i\in[k-2]$ while it has empty intersection with $H^+ - H$ otherwise.) Let σ be the face of $Q(M)$ which has nonempty intersection with both open halfspaces given by H. Then σ is the intersection of some facets of $Q(M)$ which are not flacets. Also $[\sigma\cap H, Q(M^0)](\cong [\sigma, Q(M)])$ is the Boolean algebra of order $n-1-\dim\sigma$, and hence $\Psi(Q(M^0)/(\sigma\cap H)) = \Psi(\Delta_{n-|S|})$. Since each facet of $Q(M)$ which is not a flacet corresponds to the deletion of an element of $[n]$, σ corresponds to a matroid $M|_S$ for some subset S of $[n]$. Also, σ has nonempty intersection with both open halfspaces given by H if and only if $M|_S$ has at least four parallelism classes, two of which are subsets of P_{k-1} and P_k respectively. Thus Theorem 4.4.2 implies

$$\begin{aligned}\Psi(Q(M)) =& \Psi(Q(M^+)) + \Psi(Q(M^-)) - \Psi(Q(M^0))\cdot\mathbf{c}\\ &- \sum_S \Psi(Q(M|_S))\cdot\mathbf{d}\cdot\Psi(\Delta_{n-|S|}),\end{aligned}$$

where the sum in the second line runs over all proper subsets S of $[n]$ such that $M|_S$ has at least four parallelism classes, two of which are subsets of P_{k-1} and P_k respectively. Since M^+ has $k-1$ parallelism classes, the induction hypothesis implies

$$\begin{aligned}\Psi(Q(M)) =& \sum_{i=2}^{k-1}\Psi(Q(M_i^3)) - \left(\sum_{i=2}^{k-2}\Psi(Q(M_i^2))\right)\cdot\mathbf{c}\\ &- \sum_S\left(\sum_{k=2}^{p(M|_S)-2}\Psi(Q((M|_S)_i^2))\right)\cdot\mathbf{d}\cdot\Psi(\Delta_{n-|S|}),\end{aligned}$$

where the sum in the second line runs over all proper subsets S of $[n]$ such that $M|_S$ has at least four parallelism classes, and $p(M|_S)$ is the number of parallelism classes of $M|_S$.

Since $M|_S$ corresponds to the composition

$$\beta_S := (|P_1 \cap S|, |P_2 \cap S|, \ldots, |P_k \cap S|) < \alpha$$

and there are

$$\prod_{j=1}^{k} \binom{\alpha_j}{\beta_j}$$

subsets corresponding to $\beta < \alpha$, we finish the proof. $\square$

Purtill [31] shows that the **cd**-index of the simplex Δ^n is the $(n+1)$-st André polynomial. Using the formula for the **cd**-index of a product of two polytopes given by Ehrenborg and Readdy [16], one can calculate the second and the third terms in Proposition 4.5.1. We still don't have a simple interpretation for the **cd**-index for $Q(M_\alpha)$ when α has three parts. The first few values of the **cd**-index for $Q(M_\alpha)$, where α has three parts, are displayed in Table 4.1.

α	$\Psi(Q(M_\alpha))$
$(1,1,1)$	$\mathbf{c}^2+\mathbf{d}$
$(2,1,1)$	$\mathbf{c}^3+3\mathbf{cd}+3\mathbf{dc}$
$(3,1,1)$	$\mathbf{c}^4+4\mathbf{c}^2\mathbf{d}+8\mathbf{cdc}+5\mathbf{dc}^2+7\mathbf{d}^2$
$(2,2,1)$	$\mathbf{c}^4+5\mathbf{c}^2\mathbf{d}+10\mathbf{cdc}+6\mathbf{dc}^2+10\mathbf{d}^2$
$(4,1,1)$	$\mathbf{c}^5+5\mathbf{c}^3\mathbf{d}+13\mathbf{c}^2\mathbf{dc}+15\mathbf{cdc}^2+20\mathbf{cd}^2+7\mathbf{dc}^3+18\mathbf{dcd}+22\mathbf{d}^2\mathbf{c}$
$(3,2,1)$	$\mathbf{c}^5+6\mathbf{c}^3\mathbf{d}+17\mathbf{c}^2\mathbf{dc}+20\mathbf{cdc}^2+28\mathbf{cd}^2+9\mathbf{dc}^3+26\mathbf{dcd}+33\mathbf{d}^2\mathbf{c}$
$(2,2,2)$	$\mathbf{c}^5+7\mathbf{c}^3\mathbf{d}+21\mathbf{c}^2\mathbf{dc}+24\mathbf{cdc}^2+36\mathbf{cd}^2+10\mathbf{dc}^3+34\mathbf{dcd}+42\mathbf{d}^2\mathbf{c}$
$(5,1,1)$	$\mathbf{c}^6+6\mathbf{c}^4\mathbf{d}+19\mathbf{c}^3\mathbf{dc}+29\mathbf{c}^2\mathbf{dc}^2+38\mathbf{c}^2\mathbf{d}^2+24\mathbf{cdc}^3+60\mathbf{cdcd}$
	$+72\mathbf{cd}^2\mathbf{c}+9\mathbf{dc}^4+33\mathbf{dc}^2\mathbf{d}+65\mathbf{dcdc}+47\mathbf{d}^2\mathbf{c}^2+64\mathbf{d}^3$
$(4,2,1)$	$\mathbf{c}^6+7\mathbf{c}^4\mathbf{d}+24\mathbf{c}^3\mathbf{dc}+39\mathbf{c}^2\mathbf{dc}^2+52\mathbf{c}^2\mathbf{d}^2+33\mathbf{cdc}^3+86\mathbf{cdcd}$
	$+104\mathbf{cd}^2\mathbf{c}+12\mathbf{dc}^4+48\mathbf{dc}^2\mathbf{d}+98\mathbf{dcdc}+72\mathbf{d}^2\mathbf{c}^2+100\mathbf{d}^3$
$(3,3,1)$	$\mathbf{c}^6+7\mathbf{c}^4\mathbf{d}+25\mathbf{c}^3\mathbf{dc}+42\mathbf{c}^2\mathbf{dc}^2+55\mathbf{c}^2\mathbf{d}^2+36\mathbf{cdc}^3+93\mathbf{cdcd}$
	$+114\mathbf{cd}^2\mathbf{c}+13\mathbf{dc}^4+52\mathbf{dc}^2\mathbf{d}+109\mathbf{dcdc}+81\mathbf{d}^2\mathbf{c}^2+112\mathbf{d}^3$
$(3,2,2)$	$\mathbf{c}^6+8\mathbf{c}^4\mathbf{d}+30\mathbf{c}^3\mathbf{dc}+51\mathbf{c}^2\mathbf{dc}^2+69\mathbf{c}^2\mathbf{d}^2+42\mathbf{cdc}^3+116\mathbf{cdcd}$
	$+142\mathbf{cd}^2\mathbf{c}+14\mathbf{dc}^4+64\mathbf{dc}^2\mathbf{d}+136\mathbf{dcdc}+98\mathbf{d}^2\mathbf{c}^2+142\mathbf{d}^3$

Table 4.1: **cd**-index for $Q(M_\alpha)$ for composition α with three parts

Bibliography

[1] F. Ardila and C. J. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B*, 96(1):38–49, 2006.

[2] J. Backelin and R. Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.*, 30(2):85–97, 1985.

[3] M. M. Bayer and A. Klapper. A new index for polytopes. *Discrete Comput. Geom.*, 6(1):33–47, 1991.

[4] L. J. Billera, N. Jia, and V. Reiner. A quasisymmetric function for matroids. `arXiv:math.CO/0606646`.

[5] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[6] A. Björner. The homology and shellability of matroids and geometric lattices. In *Matroid applications*, volume 40 of *Encyclopedia Math. Appl.*, pages 226–283. Cambridge Univ. Press, Cambridge, 1992.

[7] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[8] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. I. *Trans. Amer. Math. Soc.*, 348(4):1299–1327, 1996.

[9] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.

[10] A. Björner and V. Welker. The homology of "k-equal" manifolds and related partition lattices. *Adv. Math.*, 110(2):277–313, 1995.

[11] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[12] M. Davis, T. Januszkiewicz, and R. Scott. Nonpositive curvature of blow-ups. *Selecta Math. (N.S.)*, 4(4):491–547, 1998.

[13] D. S. Dummit and R. M. Foote. *Abstract algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.

[14] J. A. Eagon and V. Reiner. Resolutions of Stanley-Reisner rings and Alexander duality. *J. Pure Appl. Algebra*, 130(3):265–275, 1998.

[15] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial optimization—Eureka, you shrink!*, volume 2570 of *Lecture Notes in Comput. Sci.*, pages 11–26. Springer, Berlin, 2003.

[16] R. Ehrenborg and M. Readdy. Coproducts and the *cd*-index. *J. Algebraic Combin.*, 8(3):273–299, 1998.

[17] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.

[18] E. M. Feichtner and B. Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.

[19] I. M. Gel′fand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987.

[20] M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[21] T. H. Gulliksen and G. Levin. *Homology of local rings.* Queen's Paper in Pure and Applied Mathematics, No. 20. Queen's University, Kingston, Ont., 1969.

[22] J. Herzog, V. Reiner, and V. Welker. Componentwise linear ideals and Golod rings. *Michigan Math. J.*, 46(2):211–223, 1999.

[23] M. Khovanov. Real $K(\pi, 1)$ arrangements from finite root systems. *Math. Res. Lett.*, 3(2):261–274, 1996.

[24] C. Klivans. Shifted matroid complexes. *preprint*.

[25] D. N. Kozlov. A class of hypergraph arrangements with shellable intersection lattice. *J. Combin. Theory Ser. A*, 86(1):169–176, 1999.

[26] B. Lindström. On the realization of convex polytopes, Euler's formula and Möbius functions. *Aequationes Math.*, 6:235–240, 1971.

[27] J. L. Martin and V. Reiner. Cyclotomic and simplicial matroids. *Israel J. Math.*, 150:229–240, 2005.

[28] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[29] J. G. Oxley. *Matroid theory.* Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992.

[30] I. Peeva, V. Reiner, and V. Welker. Cohomology of real diagonal subspace arrangements via resolutions. *Compositio Math.*, 117(1):99–115, 1999.

[31] M. Purtill. André permutations, lexicographic shellability and the *cd*-index of a convex polytope. *Trans. Amer. Math. Soc.*, 338(1):77–104, 1993.

[32] V. Reiner. Lectures on matroids and oriented matroids. In *Lecture Notes for ACE Summer School in Geometric Combinatorics*. 2005.

[33] G.-C. Rota. On the combinatorics of the Euler characteristic. In *Studies in Pure Mathematics (Presented to Richard Rado)*, pages 221–233. Academic Press, London, 1971.

[34] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[35] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[36] R. P. Stanley. An introduction to hyperplane arrangements. In *Lecture Notes for IAS/PCMI Graduate Summer School*. 2004.

[37] J. W. Walker. Canonical homeomorphisms of posets. *European J. Combin.*, 9(2):97–107, 1988.

[38] D. J. A. Welsh. *Matroid theory*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976. L. M. S. Monographs, No. 8.

[39] G. M. Ziegler and R. T. Živaljević. Homotopy types of subspace arrangements via diagrams of spaces. *Math. Ann.*, 295(3):527–548, 1993.