3.1. Increasing and Decreasing Functions; Relative Extrema

Increasing and Decreasing Functions

Let \( f(x) \) be a function defined on the interval \( a < x < b \), and let \( x_1 \) and \( x_2 \) be two numbers in the interval. Then

- \( f(x) \) is increasing on the interval if \( f(x_2) > f(x_1) \) whenever \( x_2 > x_1 \).

- \( f(x) \) is decreasing on the interval if \( f(x_2) < f(x_1) \) whenever \( x_2 > x_1 \).
Procedure for using the derivative to determine intervals of increase and decrease

Step 1. Find all values of $x$ for which $f'(x) = 0$ or $f'(x)$ is not continuous, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number $c$ from each interval $a < x < b$ determined in Step 1 and evaluate $f'(c)$. Then

- If $f'(c) > 0$, $f(x)$ is increasing on $a < x < b$.
- If $f'(c) < 0$, $f(x)$ is decreasing on $a < x < b$. 

Example

Find the intervals of increase and decrease for the function

\[ f(x) = 2x^5 - 5x^4 - 10x^3 + 7 \]
Example
Find the intervals of increase and decrease for the function

\[ F(x) = \frac{x^2}{x-3}. \]
Relative Extrema

Definition

- The graph of the function \( f(x) \) is said to have a **relative maximum** at \( x = c \) if \( f(c) \geq f(x) \) for all \( x \) in an interval \( a < x < b \) containing \( c \).
- Similarly, the graph has a **relative minimum** at \( x = c \) if \( f(c) \leq f(x) \) on such an interval.
- Collectively, the relative maxima and minima of \( f \) are called its **relative extrema**.

Definition

A number \( c \) in the domain of \( f(x) \) is called a **critical number** if either \( f'(c) = 0 \) or \( f'(c) \) does not exist. The corresponding point \((c, f(c))\) on the graph of \( f(x) \) is called a **critical point** for \( f(x) \).
Relative Extrema

Relative extrema can only occur at critical points.

The First Derivative Test for Relative Extrema
Let $c$ be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is

- A relative maximum if $f'(x) > 0$ to the left of $c$ and $f'(x) < 0$ to the right of $c$.
- A relative minimum if $f'(x) < 0$ to the left of $c$ and $f'(x) > 0$ to the right of $c$.
- Not a relative extremum if $f'(x)$ has the same sign on both sides of $c$. 
Example

Find all critical numbers of the function

\[ f(x) = 2x^5 - 5x^4 - 10x^3 + 7 \]

and classify each critical point as a relative maximum, a relative minimum, or neither.
Relative Extrema

Example
Find all critical numbers of the function

$$F(x) = \frac{x^2}{x - 3}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.
Example
Find all critical numbers of the function

\[ f(x) = x\sqrt{4 - x} \]

and classify each critical point as a relative maximum, a relative minimum, or neither.
Sketching the graph

Procedure for sketching the graph of a continuous function using the derivative

Step 1. Determine the domain of $f(x)$. Set up a number line restricted to include only those numbers in the domain.

Step 2. Find $f'(x)$ and mark each critical number on the restricted number line. Then analyze the sign of $f'(x)$ to determine the intervals of increase and decrease for $f(x)$.

Step 3. For each critical number $c$, find $f(c)$ and plot the critical point $P(c, f(c))$ on a plane. Plot intercepts and other key points that can be easily found.

Step 4. Sketch the graph of $f$ as a smooth curve joining the critical points in such a way that it rises where $f'(x) > 0$, falls where $f'(x) < 0$, and has a horizontal tangent where $f'(x) = 0$. 
Example
Use calculus to sketch the graph of

\[ f(x) = 2x^5 - 5x^4 - 10x^3 + 7 \]
Example

Use calculus to sketch the graph of

\[ F(x) = \frac{x^2}{x - 3}. \]
Example

Use calculus to sketch the graph of

\[ f(x) = \frac{x + 1}{x^2 + x + 1}. \]