Math 108, Solution of Midterm Exam 3

1 It is estimated that t years from now, the population of a certain community will be

$$P(t) = -t^2 + 20t + 300$$

thousand. Use increments to estimate how much the population changes during the first month of the fifth year.

Solution. By the approximation formula with $t = 4$ and $\Delta t = \frac{1}{12}$,

$$\Delta P \approx P'(4)\Delta t = \frac{1}{4}P'(4).$$

Since $P'(t) = -2t + 20$, it follows that

$$\Delta P \approx \frac{1}{12}(-2 \cdot 4 + 20) = 1.$$

2 Find the equation of the tangent line to the curve $x^3 + y^3 = 2xy$ at the point $(1, 1)$.

Solution. Differentiating both sides of the given equation with respect to x, we get

$$\begin{aligned}
\frac{d}{dx}\left[x^3 + y^3\right] &= \frac{d}{dx}\left[2xy\right] \\
3x^2 + 3y^2\frac{dy}{dx} &= 2y\frac{d}{dx}(x) + 2x\frac{d}{dx}(y) \\
3x^2 + 3y^2\frac{dy}{dx} &= 2y + 2x\frac{dy}{dx} \\
(3y^2 - 2x)\frac{dy}{dx} &= 2y - 3x^2 \\
\frac{dy}{dx} &= \frac{2y - 3x^2}{3y^2 - 2x}.
\end{aligned}$$

So, the slope of the tanget line is

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = -1$$

and hence the equation of the tangent line is

$$\begin{aligned}
y - 1 &= -1(x - 1) \\
y &= -x + 2
\end{aligned}$$

3 When the price of a certain commodity is p dollars per unit, the manufacturer is willing to supply x hundred units, where

$$p^2 - 4x^2 = 9.$$

How fast is the supply changing when the price is \$5 per unit and is decreasing at the rate of 40 cents per month?

Solution. We want to find $\frac{dx}{dt}$ when $p = 5$ and $\frac{dp}{dt} = -0.4$. Differentiating the equation $p^2 - 4x^2 = 9$ implicitly with respect to time, we get

$$2p\frac{dp}{dt} - 8x\frac{dx}{dt} = 0$$

so that

$$-8x\frac{dx}{dt} = -2p\frac{dp}{dt}$$

and

$$\frac{dx}{dt} = \frac{p}{4x}\frac{dp}{dt}$$

When $p = 5$, the supply satisfies

$$\begin{aligned} 5^2 - 4x^2 &= 9 \\ 4x^2 &= 16 \\ x^2 &= 4 \\ x &= 2 \end{aligned}$$

and by substituting $p = 5$, $x = 2$, and $\frac{dp}{dt} = -0.4$ into the formula for $\frac{dx}{dt}$, we obtain

$$\frac{dx}{dt} = \frac{5}{4 \cdot 2}(-0.4) = -\frac{1}{4} = -0.25$$

hundred units per month.

4 Let f be a function defined by $f(x) = x^4 - 4x^3 + 3$.

(a) Find intervals on which f is increasing and decreasing.

Solution. The derivative of f is

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x^4 - 4x^3 + 3\right] \\ &= 4x^3 - 12x^2 \\ &= 4x^2(x - 3) \end{aligned}$$

which is continuous everywhere, with $f'(x) = 0$ where $x = 0$ and $x = 3$. The numbers 0 and 3 divide the x axis into three open intervals; namely, $x < 0$, $0 < x < 3$, and $x > 3$. Choose a test number from each of these intervals; say $c = -1$ from $x < 0$, $c = 1$ from $0 < x < 3$, and $c = 4$ from $x > 3$. Then, evaluate $f'(c)$ for each test number:

$$\begin{aligned} f'(-1) &= 4(-1)^2(-1-3) = -12 < 0, \\ f'(1) &= 4(1)^2(1-3) = -8 < 0, \\ f'(4) &= 4(4)^2(4-3) = 64 > 0. \end{aligned}$$

So, $f(x)$ is decreasing on $x < 0$ and $0 < x < 3$ and increasing on $x > 3$.

(b) Find all critical points of f and classify each of them as a relative maximum, a relative minimum, or neither.

Solution. Since $f'(x)$ exists for all x, the only critical numbers are where $f'(x) = 0$, that is, $x = 0$ and $x = 3$. Corresponding critical points are $(0, 3)$ and $(3, 3^4 - 4(3)^3 + 3) = (3, -24)$. Since $f'(x)$ has the same sign (minus) on both sides of $x = 0$, the critical point $(0, 3)$ is not a relative extremum. Since $f'(x) < 0$ to the left of $x = 3$ and $f'(x) > 0$ to the right of $x = 3$, the critical point $(3, -24)$ is a relative minimum.

(c) Find intervals on which the graph of f is concave up and concave down.

Solution. Since the first derivative of f is

$$f'(x) = 4x^3 - 12x^2,$$

the second derivative of f is

$$\begin{aligned} f''(x) &= 12x^2 - 24x \\ &= 12x(x-2) \end{aligned}$$

The second derivative $f''(x)$ is continuous for all x and $f''(x) = 0$ for $x = 0$ and $x = 2$. These numbers divide the x axis into three intervals; namely $x < 0$, $0 < x < 2$, and $x > 2$. Evaluating $f''(x)$ at test numbers in each of these intervals (say, $x = -1$, $x = 1$, and $x = 3$, respectively), we find

$$\begin{aligned} f''(-1) &= 12(-1)(-1-2) > 0, \\ f''(1) &= 12(1)(1-2) < 0, \\ f''(3) &= 12(3)(3-2) > 0. \end{aligned}$$

Thus, the graph of $f(x)$ is concave up for $x < 0$ and $x > 2$ and concave down for $0 < x < 2$.

(d) Find all inflection points of f.

Solution. Since the concavity changes from upward to downward at $x = 0$ and $f(0) = 3$, it follows that $(0, 3)$ is an inflection point. Since the concavity changes from downward to upward at $x = 2$ and $f(2) = (2)^4 - 4(2)^3 + 3 = -13$, $(2, -13)$ is another inflection point.

(e) Sketch the graph of f.

5 Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{2x^2 - 5x + 3}{x^2 + 1}$.

Solution. Since $x^2 + 1 \neq 0$, the given function is continuous everywhere. Thus, there are no vertical asymptotes.

Dividing numerator and denominator by x^2, we find that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{2x^2 - 5x + 3}{x^2 + 1} = \lim_{x\to+\infty} \frac{2 - \dfrac{5}{x} + \dfrac{3}{x^2}}{1 + \dfrac{1}{x^2}} = \frac{2 - 0 + 0}{1 + 0} = 2$$

and similarly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} \frac{2x^2 - 5x + 3}{x^2 + 1} = \lim_{x\to-\infty} \frac{2 - \dfrac{5}{x} + \dfrac{3}{x^2}}{1 + \dfrac{1}{x^2}} = \frac{2 - 0 + 0}{1 + 0} = 2$$

Thus, the graph of $f(x)$ has $y = 2$ as a horizontl asymptote.

6 Find the absolute maximum and absolute minimum (if any) of the function $f(x) = x^4 - 8x^2 + 10$ on the interval $-1 \leq x \leq 3$.

Solution. From the derivative

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2)$$

we see that the critical numbers are $x = -2$, $x = 0$, and $x = 2$. Of these, $x = 0$ and $x = 2$ lie in the interval $-1 < x < 3$. Compute $f(x)$ at the critical numbers $x = 0$ and $x = 2$ and at endpoints $x = -1$ and $x = 3$.

$$f(0) = 0^4 - 8(0)^2 + 10 = 10$$
$$f(2) = 2^4 - 8(2)^2 + 10 = 16 - 32 + 10 = -6$$
$$f(-1) = (-1)^4 - 8(-1)^2 + 10 = 1 - 8 + 10 = 3$$
$$f(3) = 3^4 - 8(3)^2 + 10 = 81 - 72 + 10 = 19$$

Compare these values to conclude that the absolute maximum of $f(x)$ on the interval $-1 \leq x \leq 3$ is $f(3) = 19$ and the absolute minimum is $f(2) = -6$.

7 A manufacturer estimates that when q units of a particular commodity are produced each month, the total cost will be $C(q) = q^2 - 12q + 25$ dollars and all q units can be sold at a price of $p(q) = 20 - q$ dollars per unit. At what level of production is the profit maximized?

Solution. The revenue is

$$R(q) = q \cdot p(q) = q(20 - q) = 20q - q^2$$

dollars, so the profit is

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= (20q - q^2) - (q^2 - 12q + 25) \\ &= -2q^2 + 32q - 25. \end{aligned}$$

Since

$$P'(q) = -4q + 32 = -4(q - 8),$$

$q = 8$ is the only critical number of $P(q)$ on $q > 0$. Since $P''(q) = -4$, it follows that $P''(8) < 0$. Thus the maximum profit is achieved when $q = 8$.