Math 108, Solution of Midterm Exam 1

1 What is the domain of each of the following functions?

(a) $f(x) = \dfrac{x^2 + 4x + 3}{x^2 + x - 2}$

Solution. Since division by any nonzero number is possible, the domain of f is the set of all numbers satisfying $x^2 + x - 2 \neq 0$. Since $x^2 + x - 2 = (x-1)(x+2)$, the domain of f is the set of all real numbers x except $x = 1$ and $x = -2$.

(b) $g(x) = x^2 - 4 + \sqrt{1 - x^2}$

Solution. Since negative numbers do not have real square roots, $g(x)$ can be defined only when $1 - x^2 \geq 0$, i.e., $x^2 - 1 \leq 0$. Since $x^2 - 1 = (x-1)(x+1)$, the domain of g is the set of all real numbers x such that $-1 \leq x \leq 1$.

2 Market research indicates that consumers will buy x units of a particular kind of camera when the unit price is

$$p = 500 - x$$

dollars. Total cost consists of a fixed overhead of \$30,000 plus production cost of \$100 apiece.

(a) Express the manufacturer's monthly profit as a function of the level of production x.

Solution. The total revenue is

$$\begin{aligned} R(x) &= (\text{price per camera})(\text{number of cameras sold}) \\ &= (500 - x)x \end{aligned}$$

Also, the total cost for producing x cameras is

$$\begin{aligned} C(x) &= (\text{fixed overhead}) + (\text{production cost per unit})(\text{number of cameras sold}) \\ &= 30,000 + 100(x) \end{aligned}$$

Thus the profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (500 - x)x - (30,000 + 100x) \\ &= -x^2 + 400x - 30,000 \end{aligned}$$

(b) For what values of x is production of the cameras profitable?

Solution. Production is profitable when $P(x) > 0$. Since

$$\begin{aligned} P(x) = -x^2 + 400x - 30,000 &> 0 \\ x^2 - 400x + 30,000 &< 0 \\ (x-100)(x-300) &< 0 \end{aligned}$$

production is profitable for $100 < x < 300$.

3 Find all values of x such that $f(g(x)) = g(f(x))$, where $f(x) = \frac{1}{x}$ and $g(x) = \frac{2+x}{4-x}$.

Solution. Replace x by $g(x) = \frac{2+x}{4-x}$ in the formula for $f(x)$ to get

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{\frac{2+x}{4-x}} = \frac{4-x}{2+x}.$$

Similarly, we replace x by $f(x) = \frac{1}{x}$ in the formula for $g(x)$ to get

$$g(f(x)) = \frac{2+\frac{1}{x}}{4-\frac{1}{x}} = \frac{2x+1}{4x-1}.$$

By solving the equation $f(g(x)) = g(f(x))$, we get

$$\begin{aligned} \frac{4-x}{2+x} &= \frac{2x+1}{4x-1} \\ (4-x)(4x-1) &= (2x+1)(2+x) \\ 16x - 4 - 4x^2 + x &= 4x + 2x^2 + 2 + x \\ -6x^2 + 12x - 6 &= 0 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Thus $f(g(x)) = g(f(x))$ only when $x = 1$.

4 Find the slope-intercept form of the equation for each of the following lines:

(a) Line L_1 that passes through $(-1, 2)$ and $(2, -1)$

Solution. The slope is

$$m_1 = \frac{(-1) - 2}{2 - (-1)} = \frac{-3}{3} = -1.$$

Then use the point-slope formula with $(-1, 2)$ to get

$$y - 2 = -1[x - (-1)].$$

Thus the slope-intercept form of the equation of the given line is

$$y = -x + 1.$$

(b) Line L_2 that passes through $(1, 5)$ and is perpendicular to the line $L_3 : 2x + 4y = 3$.

Solution. By rewriting the equation $2x + 4y = 3$ in the slope-intercept form $y = -\frac{1}{2}x + \frac{3}{4}$, we see that the line L_3 has slope $m_3 = -\frac{1}{2}$. Thus the line L_2 has slope $m_2 = -\frac{1}{m_3} = 2$. Since the line L_2 contains $(1, 5)$, we have

$$\begin{aligned} y - 5 &= 2(x - 1) \\ y &= 2x + 3 \end{aligned}$$

5 Find all x and y intercepts of the function $f(x) = x^2 - 2x - 8$.

Solution. Since $f(0) = -8$, the y intercept is $(0, -8)$. To find the x intercepts, solve the equation $f(x) = 0$. Factoring, we find that

$$\begin{aligned} x^3 - 2x - 8 &= 0 \\ (x + 2)(x - 4) &= 0 \end{aligned}$$

Thus, the x intercepts are $(-2, 0)$ and $(4, 0)$.

6 Find all points of intersection of the graphs of $f(x) = x^2 - 2x + 2$ and $g(x) = 2x - 1$.

Solution. You must solve the equation $f(x) = g(x)$. Since

$$\begin{aligned} x^2 - 2x + 2 &= 2x - 1 \\ x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \end{aligned}$$

the solutions are $x = 1$ and $x = 3$. Computing the corresponding y coordinates from the equation $y = 2x - 1$, the points of intersections are $(1, 1)$ and $(3, 5)$.

7 Find each of the following limits or show it does not exist. If the limit is infinite, indicate whether it is $+\infty$ or $-\infty$.

(a) $\displaystyle\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - x - 2}$

Solution. As $x \to 2$, both the numerator and the denominator approach zero. Since the numerator and the denominator have a common factor $x - 2$, we have

$$\begin{aligned} \lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - x - 2} &= \lim_{x\to 2} \frac{(x-2)(3x+1)}{(x-2)(x+1)} \\ &= \lim_{x\to 2} \frac{3x+1}{x+1} \\ &= \frac{3 \cdot 2 + 1}{2 + 1} = \frac{7}{3}. \end{aligned}$$

(b) $\displaystyle\lim_{x\to -\infty} \frac{x^3 - x + 2}{x^2 - x - 2}$

Solution. The highest power in the denominator is x^2. Divide the numerator and the demonimator by x^2 to get

$$\lim_{x\to -\infty} \frac{x^3 - x + 2}{x^2 - x - 2} = \lim_{x\to -\infty} \frac{x - 1/x + 2/x^2}{1 - 1/x - 2/x^2}$$

Since

$$\lim_{x\to -\infty} \left(x - 1/x + 2/x^2\right) = -\infty \quad \text{and} \quad \lim_{x\to -\infty} \left(1 - 1/x - 2/x^2\right) = 1$$

it follows that

$$\lim_{x\to -\infty} \frac{x^3 - x + 2}{x^2 - x - 2} = -\infty$$

(c) $\displaystyle\lim_{x\to 1^+} \frac{x - 2}{x^2 + x - 2}$

Solution. By factoring, we can see

$$\frac{x-2}{x^2+x-2} = \frac{x-2}{(x+2)(x-1)}$$

near $x = 1$. The numerator $x - 2$ approaches $1 - 2 = -1$ as x approaches 1 from the right. Since $x + 2$ approaches $1 + 2 = 3$ and $x - 1$ approaches 0 and is always positive as x approaches 1 from the right, the denominator approaches 0 and is always positive as x approaches 1 from the right. Thus the function $\dfrac{x-2}{x^2+x-2}$ decreases without bound, i.e.,

$$\lim_{x\to 1^+} \frac{x-2}{x^2+x-2} = -\infty.$$

8 List all values of x for which the function $f(x)$ is not continuous, where

$$f(x) = \begin{cases} \dfrac{x^2+1}{x-1} & \text{if } x < 0 \\ x^2 - 3x + 4 & \text{if } 0 \leq x < 2 \\ 2x - 2 & \text{if } x \geq 2 \end{cases}$$

Solution. Since $\dfrac{x^2+1}{x-1}$ is a rational function defined everywhere except at $x = 1$ and $x^2 - 3x + 4$ and $2x - 2$ are polynomials defined everywhere, $f(x)$ is continuous everywhere possibly except $x = 0$ and at $x = 2$. For $x = 0$, we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+1}{x-1} = \frac{0^2+1}{0-1} = \frac{1}{1} = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2 - 3x + 4) = 0^2 - 3(0) + 4 = 4$$

Thus $\lim_{x\to 0} f(x)$ does not exist, and so the function $f(x)$ is not continuous at $x = 0$.

Since

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 3x + 4) = 2^2 - 3(2) + 4 = 2$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x - 2) = 2(2) - 2 = 2,$$

we have $\lim_{x\to 2} f(x) = 2$. Since $f(2) = 2(2) - 2 = 2$, $f(x)$ is continuous at $x = 2$. Therefore, $f(x)$ is not continuous only at $x = 0$.

9 Find the value of the constant A that makes the function $f(x)$ continuous for all x, where

$$f(x) = \begin{cases} Ax + 4 & \text{if } x < 2 \\ x^2 + x + 2 & \text{if } x \geq 2 \end{cases}$$

Solution. Since $Ax + 4$ and $x^2 + x + 2$ are both polynomials, $f(x)$ is continuous everywhere except possibly at $x = 2$. We have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (Ax + 4) = 2A + 4$$

and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + x + 2) = (2)^2 + 2 + 2 = 8.$$

Thus, for $\lim_{x \to 2} f(x)$ to exist, we must have $2A + 4 = 8$ or $A = 2$, in which case

$$\lim_{x \to 2} f(x) = 8 = f(2).$$

Thus f is continuous for all x only when $A = 2$.