

FOURIER ANALYSIS

SEAN P. CARNEY

CONTENTS

1. Part 1: least-squares approximation and L^2 convergence	1
1.1. Sequences and series of functions	1
1.2. Discrete least squares problem	3
1.3. L^2 : inner product space of square integrable functions	4
1.4. Trigonometric polynomials	5
1.5. Parseval's relation	6
2. Part 2: Summary of main ideas	8
2.1. Important properties of a function's Fourier series	10
3. Part 3: Discrete Fourier Transform (DFT)	12
3.1. Fast Fourier Transform:	13
3.2. Additional exercises	13
4. Part 4: Pointwise convergence of Fourier series & Gibbs phenomenon	15
5. Part 5: Fourier Transform on the real line \mathbb{R}	19
5.1. Additional Exercises	22

Most of the content adapted from Chapter 8 of Peter Duren's "Invitation to Classical Analysis".

1. PART 1: LEAST-SQUARES APPROXIMATION AND L^2 CONVERGENCE

1.1. Sequences and series of functions.

Definition 1.1 (Uniform convergence of functions). Let $f : [a, b] \rightarrow \mathbb{R}$, and let $\{f_n\}$ be a sequence of functions each mapping from $[a, b] \rightarrow \mathbb{R}$. We say that $\{f_n\}$ is *uniformly convergent* to f on $[a, b]$ if $\forall \epsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

whenever $n \geq N$.

Note that uniform convergence *implies* pointwise convergence, but not necessarily the other way around. The key difference is that in uniform convergence, the number $N = N(\epsilon)$ does *not depend on x* .

Exercise 1.2 (Uniform limit of continuous functions). Suppose that $\{f_n\}$ is a sequence of *continuous* functions, each mapping from $[a, b] \rightarrow \mathbb{R}$. Suppose that $f_n(x) \rightarrow f(x)$ *uniformly* on $[a, b]$. Show that f is continuous on $[a, b]$.

Exercise 1.3 (Counter-example). Can you give an example of a sequence of continuous functions $\{f_n\}$ that converges *only* pointwise (i.e. *not* uniformly) to a discontinuous function on $[0, 1]$?

Corollary 1.4 (Uniformly convergent series). Let $s_n(x) = \sum_{k=1}^n u_k(x)$ and suppose that each $u_k(x)$ is continuous, so that $s_n(x)$ is also continuous (since finite sums of continuous functions are continuous). If the sequence $\{s_n\}$ is uniformly convergent to some s on $[a, b]$, then the infinite series $\sum_{k=1}^{\infty} u_k(x)$ is also continuous.

Note that uniform convergence also allows us to differentiate/integrate our series term by term, so that

$$\int_a^b s(x) dx = \int_a^b \sum_{k=1}^{\infty} u_k(x) dx = \sum_{k=1}^{\infty} \int_a^b u_k(x) dx$$

and

$$\frac{d}{dx} s(x) = \frac{d}{dx} \sum_{k=1}^{\infty} u_k(x) = \sum_{k=1}^{\infty} \frac{d}{dx} u_k(x),$$

provided, of course, that the functions s and each u_k are integrable/differentiable.

The following theorem states that any continuous function can be approximated *uniformly* by algebraic polynomials of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n.$$

Theorem 1.5 (Weierstrass Approximation Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function, where $[a, b]$ is closed and bounded (we'll denote this by saying $f \in C([a, b])$). Then $\forall \epsilon > 0$ there exists a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon, \quad \forall x \in [a, b].$$

It is a classic theorem and worth proving, but we'll instead elect to later prove an analogous theorem for *trigonometric* polynomials—see Exercise 1.19.

1.2. Discrete least squares problem. First, determine a least squares approximation in a relatively simple setting:

Exercise 1.6 (Discrete linear least-squares problem). Let $\{(x_i, y_i)\}_{i=1}^N$ be a collection of points in \mathbb{R}^2 , where each x_i is distinct. The linear least-squares approximation is defined to be $L(x) = a_0 + a_1x$, where a_0 and a_1 are chosen to *minimize*

$$E(a_0, a_1) = \sum_{i=1}^N (y_i - L(x_i))^2.$$

Determine a_0 and a_1 that define the linear least-squares approximation.

This next exercise is an application of the one just completed; it's optional/just for fun.

Exercise 1.7 (Population predictor using linear least-squares). Consider the data in the table below which shows the population in the US (in thousands) for the last six censuses.

Year	1960	1970	1980	1990	2000	2010
Population (in thousands)	179,323	203,302	226,542	249,633	281,422	308,746

begin equation bigskip

(a) Use your answer to Exercise 1.6 to construct a linear least-squares approximation $L(x)$ to the data. It is convenient to let x equal the number of years since 1960.

(b) Use $L(x)$ to predict the population of the US in 2020 (in thousands).

(c) The 2020 census recorded a population of 331,449,281 people. How close is the real number to the approximation from part (b)?

(d) What does the model predict for 2040?

1.3. L^2 : inner product space of square integrable functions.

Definition 1.8 (L^2 function space and inner product). For two functions $f, g : [a, b] \rightarrow \mathbb{R}$, define the L^2 inner product

$$\langle f, g \rangle := \int_a^b f(x)g(x) dx.$$

f and g are said to be *orthogonal* functions if $\langle f, g \rangle = 0$.

Additionally, the inner product $\langle \cdot, \cdot \rangle$ can be used to define the L^2 norm:

$$\|f\|_{L^2([a,b])} := \langle f, f \rangle^{1/2}.$$

We say that $f \in L^2([a, b])$ whenever $\|f\|_{L^2([a,b])} < \infty$.

Remark 1.9. The L^2 space generalizes the concept of the Euclidean norm for vectors $x \in \mathbb{R}^n$ (a finite dimensional vector space), given by

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2},$$

to functions, which can be viewed as infinite dimensional vectors. The L^2 space is an important example of a *Hilbert space*, an important object of study in functional analysis.

The following calculus identities will be used extensively in our development of the Fourier series.

Exercise 1.10 (*Orthogonality relations*). Let $[a, b] = [-\pi, \pi]$. Show that:

- (a) $\langle \cos(nx), K \rangle = \langle \sin(nx), K \rangle = 0$ for any constant $K \in \mathbb{R}$ and any $n = 1, 2, 3, \dots$
- (b) $\langle \cos(nx), \sin(mx) \rangle = 0$ for all $n, m = 1, 2, 3, \dots$
- (c) $\langle \cos(nx), \cos(mx) \rangle = \langle \sin(nx), \sin(mx) \rangle = 0$ for $n \neq m$.
- (d) $\langle \cos(nx), \cos(nx) \rangle = \langle \sin(nx), \sin(nx) \rangle = \pi$ for all $n = 1, 2, 3, \dots$

Hint: it could be useful to utilize Euler's formula: $e^{ix} = \cos(x) + i \sin(x)$, which implies

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

1.4. Trigonometric polynomials. Next we define the building blocks of a function's Fourier series.

Definition 1.11 (Trigonometric polynomial). For some positive integer n , define

$$(1) \quad T_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$$

to be a trigonometric polynomial of degree n .

Exercise 1.12 (Continuous least squares). Let $f \in L^2([-\pi, \pi])$, and let $n > 0$. Determine the coefficients $\{a_0, a_1, \dots, a_n, b_1, b_2, \dots, b_n\}$ that ensure the L^2 "error" E between f and T_n

$$E(a_0, a_1, \dots, a_n, b_1, \dots, b_n) = \|f - T_n\|_{L^2([-\pi, \pi])}^2 = \int_{-\pi}^{\pi} (f(x) - T_n(x))^2 dx$$

is minimized.

Hint: Write

$$\|f - T_n\|_{L^2([-\pi, \pi])}^2 = \|f\|_{L^2[-\pi, \pi]}^2 - 2\langle f, T_n \rangle + \|T_n\|_{L^2[-\pi, \pi]}^2$$

Then the orthogonality relations ought to make life relatively simple.

Later on we will call the coefficients just computed the Fourier coefficients of a function f .

Exercise 1.13. [Precursor to Bessel's inequality] Let T_n be defined by (1) with coefficients given by the minimizers found in Exercise 1.12. Prove that

$$(2) \quad \frac{1}{2}\pi a_0^2 + \pi \sum_{k=1}^n (a_k^2 + b_k^2) \leq \|f\|_{L^2([-\pi, \pi])}^2.$$

Hint: use (i) that $\|f - T_n\|_{L^2}^2 = \int (f - T_n)^2 dx \geq 0$, (ii) the orthogonality relations, and (iii) your answer from Exercise 1.12.

Exercise 1.14 (Bessel's inequality). Let $f \in L^2([-\pi, \pi])$. Then the limit of the LHS of the inequality in (2) exists as $N \rightarrow \infty$. Why?

The resulting inequality

$$(3) \quad \frac{1}{2}\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \|f\|_{L^2([-\pi, \pi])}^2.$$

is called *Bessel's inequality*.

Remark 1.15 (Bessel's inequality). The derivation of Bessel's inequality here was for trigonometric polynomials. The inequality is valid, however, for more general classes of functions; only the orthogonality property is needed.

This final section is heavier on concepts useful to know in real analysis.

1.5. Parseval's relation. For any square-integrable function $f \in L^2([-\pi, \pi])$, the inequality (3) is actually an *equality*, meaning f can be represented as a trigonometric polynomial of infinite degree. The result is called *Parseval's relation*, which we develop now as a consequence of two important results:

- the Weierstrass approximation theorem for the case of trigonometric polynomials
- approximation of an $L^2([-\pi, \pi])$ function by convolution with an *approximation to the identity*

We'll state the two results first. They'll then be utilized below in Theorem 1.18, and their proofs are assigned in Exercises 1.19 and 1.20 below.

Theorem 1.16 (Weierstrass approximation theorem for trigonometric polynomials). *Let $f \in C([-\pi, \pi])$ and periodic, so that $f(-\pi) = f(\pi)$. Then $\forall \epsilon > 0$, there exists a trigonometric polynomial of some degree m such that*

$$|f(x) - T_m(x)| < \epsilon, \quad \forall x \in [-\pi, \pi].$$

Theorem 1.17 (Approximation of an L^2 function by a continuous periodic function). *Let $f \in L^2([-\pi, \pi])$. Then $\forall \epsilon > 0 \exists g \in C([-\pi, \pi])$ such that $g(-\pi) = g(\pi)$ and*

$$\|f - g\|_{L^2([-\pi, \pi])} < \epsilon.$$

Proof. The general idea is to construct such a function g by the use of convolution against a smooth (i.e. C^∞), periodic function φ_δ whose integral $\int \varphi_\delta = 1$ for all $\delta > 0$ and whose support is only on $[-\delta, \delta]$. Such a function is called an approximation to the identity.

□

We can now prove our main theorem of interest.

Theorem 1.18 (*L^2 convergence & Parseval's relation*). Let $f \in L^2([-\pi, \pi])$. For fixed $n > 0$, let T_n be the least-squares trigonometric approximation to f , that is, let the coefficients of T_n be the minimizers found in Exercise 1.12. Then

$$(4) \quad \lim_{n \rightarrow \infty} \|f - T_n\|_{L^2([-\pi, \pi])} = \lim_{n \rightarrow \infty} \left(\int_{-\pi}^{\pi} (f(x) - T_n(x))^2 dx \right)^{1/2} = 0$$

which also implies

$$(5) \quad \|f\|_{L^2([-\pi, \pi])}^2 = \int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{2}\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

Proof. To show (4), i.e. convergence in the $L^2([-\pi, \pi])$ norm, we need to show $\exists N > 0$ such that for arbitrary $\epsilon > 0$ and all $n \geq N$,

$$\|f - T_n\|_{L^2([-\pi, \pi])} = \left(\int_{-\pi}^{\pi} (f(x))^2 dx \right)^{1/2} < \epsilon.$$

We'll try an $\epsilon/2$ -type argument; first we'll use Theorem 1.17, which will in turn let us use the Weierstrass theorem.

Fix $\epsilon > 0$. By Theorem 1.17, $\exists g$ that is both periodic and continuous on $[-\pi, \pi]$ that satisfies $\|f - g\|_{L^2([-\pi, \pi])} < \epsilon/2$. The Weierstrass approximation theorem can then be applied to approximate g : $\exists M > 0$ such that the trigonometric polynomial T_M satisfies $|g(x) - T_M(x)| < (\epsilon/2)^2/2\pi$ for every $x \in [-\pi, \pi]$. Then

$$\|g - T_M\|_{L^2([-\pi, \pi])} = \left(\int_{-\pi}^{\pi} (g(x) - T_M(x))^2 dx \right)^{1/2} < ((\epsilon/2)^2)^{1/2} = \epsilon/2$$

A simple addition/subtraction and application of the triangle inequality then gives

$$\|f - T_M\|_{L^2([-\pi, \pi])} \leq \|f - g\|_{L^2([-\pi, \pi])} + \|g - T_M\|_{L^2([-\pi, \pi])} < \epsilon/2 + \epsilon/2 = \epsilon$$

Finally, whenever $n \geq M$, the fact that T_n is the least-squares trigonometric approximation to f implies

$$\|f - T_n\|_{L^2([-\pi, \pi])} \leq \|f - T_M\|_{L^2([-\pi, \pi])} < \epsilon,$$

which gives the result. □

Exercise 1.19 (Proof of trigonometric form of Weierstrass approximation).

Exercise 1.20 (Proof approximation in L^2 by continuous function).

2. PART 2: SUMMARY OF MAIN IDEAS

To begin we'll officially define a function's Fourier coefficients.

Definition 2.1 (Fourier coefficients). Let $f \in L^2([-\pi, \pi])$. For each integer $n \geq 0$ define the Fourier coefficients of f to be

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

(note $b_0 = 0$). Using the notation of the L^2 inner product, these are simply

$$a_n = \frac{1}{\pi} \langle f(x), \cos(nx) \rangle, \quad b_n = \frac{1}{\pi} \langle f(x), \sin(nx) \rangle.$$

We've shown that if $f \in L^2([-\pi, \pi])$, then there exists a sequence $\{T_n\}$ of trigonometric polynomials

$$T_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$$

that converges in $L^2([-\pi, \pi])$ whenever the coefficients $\{a_0, a_1, \dots, a_n, b_1, \dots, b_n\}$ are defined as in Definition 2.1. This means:

$$\lim_{n \rightarrow \infty} \|f - T_n\|_{L^2([-\pi, \pi])} = 0.$$

Moreover, the L^2 "energy" of f is given by an infinite sum of its Fourier coefficients squared, up a factor of π :

$$\|f\|_{L^2([-\pi, \pi])}^2 = \pi \left(\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right).$$

In this way, a satisfying way to view what we've just shown is that a function's Fourier series is a map from square integrable functions L^2 on $[-\pi, \pi]$ to *square-summable* sequences $(a_0, a_1, b_1, a_2, b_2, \dots)$.

Remark 2.2 (Equivalent formulation with complex exponentials). In this view, it is not uncommon to write a function's Fourier series in terms of the *complex exponential*, instead of sines/cosines. The two formulations are entirely equivalent. Instead of T_n , we could write

$$U_n = \sum_{n=-N}^N c_n e^{inx}.$$

Using Euler's formula (that $e^{i\theta} = \cos(\theta) + i \sin(\theta)$), we can easily show that if

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n) = \overline{c_n},$$

then $U_n = T_n$, where $\bar{\cdot}$ denotes the complex conjugate. This means the Fourier coefficients for a L^2 function f are

$$(6) \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

and the L^2 convergence and Parseval relation become

$$\lim_{n \rightarrow \infty} \|f - U_n\|_{L^2([-\pi, \pi])} = 0, \quad \|f\|_{L^2([-\pi, \pi])}^2 = \pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

where $|\cdot|$ is the complex modulus.

In this way, the Fourier series is a map from

$$L^2([-\pi, \pi]) \rightarrow l^2(\mathbb{Z}),$$

where a sequence $\{\alpha_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ if and only if $\sum_{k \in \mathbb{Z}} |\alpha_k|^2 < \infty$.

2.1. Important properties of a function's Fourier series. There are two properties of the Fourier series that we'll highlight. We'll rely on the complex interpretation of the Fourier series as a map from square-integrable functions to square-integrable sequences.

Definition 2.3 (Operator notation \mathcal{F}). Let $\mathcal{F} : L^2(-[\pi, \pi]) \rightarrow l^2(\mathbb{Z})$ denote the map that takes $f \in L^2(-[\pi, \pi])$ and maps to $\{c_k\}_{k \in \mathbb{Z}}$:

$$(\mathcal{F}(f))_k = c_k,$$

where c_k are the Fourier coefficients (6).

The first property is \mathcal{F} maps differentiation to multiplication by a polynomial. The crucial implication is that one can relate the rate of decay of a function's Fourier coefficients to its smoothness (i.e. how many times it can be differentiated).

The second is that \mathcal{F} maps multiplication to convolution. This fact has led to the successful development of Fourier analysis for compact groups (see the *Peter-Weyl theorem*) and locally compact Abelian groups (see the notion of *Pontryagin duality*).

Exercise 2.4 (Derivative property). Let $f \in L^2(-[\pi, \pi])$ with $f(-\pi) = f(\pi)$. Suppose that additionally $f \in C^p(-[\pi, \pi])$ for some integer $p > 0$, so that f is p times differentiable, and its p -th derivative is continuous.

(a) Show that the k -th Fourier coefficient of df/dx equals ikc_k , i.e. that $\mathcal{F}(df/dx) = \{ikc_k\}_{k \in \mathbb{Z}}$. Conclude then that

$$\mathcal{F}\left(\frac{d^p f}{dx^p}\right) = \{(ik)^p c_k\}_{k \in \mathbb{Z}}.$$

Hint: use integration by parts.

(b) If a function is continuous on a closed and bounded interval, it is necessarily square-integrable there. Thus, $f \in C^p(-[\pi, \pi]) \implies f^{(p)} \in L^2(-[\pi, \pi])$. Additionally, we know that if a sequence $\{\alpha_n\}_{n \in \mathbb{Z}}$ satisfies

$$\sum_{n \in \mathbb{Z}} \alpha_n < \infty,$$

then it necessarily implies that $|\alpha_n| \rightarrow 0$ as $n \rightarrow \infty$.

With these facts in mind, use Parseval's relation and part (a) to show that the Fourier coefficients of f satisfy

$$\lim_{k \rightarrow \infty} |k|^p |c_k| = 0.$$

Remark 2.5. The differentiability property means that the smoothness, or degree of regularity of a function, is directly related to the rate of decay of its Fourier coefficients. In particular, if the coefficients decay *exponentially*, i.e. faster than $|k|^p$ for *any* $p > 0$, that means a function is infinitely differentiable. A special case is when $c_k = 0 \forall |k| > K$ for some fixed $K > 0$. A function f that satisfies this property is called *band-limited*.

Definition 2.6 (Convolution). Let $f, g \in L^2(-[\pi, \pi])$ and let f and g both be 2π periodic. Define a new function $(f * g)(x)$ by

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-t)g(t) dt$$

(note: although $x-t$ could take values outside of $[-\pi, \pi]$, this is well defined by periodicity of f).

Exercise 2.7 (Convolution property). Let $f, g \in L^2([-\pi, \pi])$ and let f and g both be 2π periodic. Show then that up to a factor of 2π , the k -th Fourier coefficient of the convolution $f * g$ is equivalent to the two k -th Fourier coefficients of f and g multiplied together:

$$(7) \quad \mathcal{F}(f * g)_k = 2\pi \mathcal{F}(f)_k \mathcal{F}(g)_k.$$

Hint: First write what the left hand side of (7) means by definition. Then use Fubini's theorem to exchange the order of integration (this is justified since $f, g \in L^2$), and use the fact that for any 2π periodic function h , $\int_{-\pi}^{\pi} h(t) dt = \int_{-\pi+s}^{\pi+s} h(t) dt$ for any $s \in \mathbb{R}$.

Remark 2.8. As mentioned above, a theory of Fourier series on compact and locally compact Abelian groups has been developed based on preserving this property. Investigate the Peter-Weyl theorem and Pontryagin duality for further reading.

3. PART 3: DISCRETE FOURIER TRANSFORM (DFT)

As stressed in “Part 2: Summary of main ideas”, what is often called a function’s “Fourier series” is a map from square integrable functions to square-summable sequences, that is, from $L^2([-\pi, \pi]) \rightarrow l^2(\mathbb{Z})$. This map is from one infinite dimensional space to another; functions can be viewed as infinite dimensional ‘vectors’, and by definition sequences on \mathbb{Z} consist of a countably infinite number of entries.

There exists an analogous map whose domain and range are *finite* dimensional sequences, i.e. vectors. It’s usually called the *discrete Fourier transform* (DFT) and it is a tool with great utility in an impressive number of fields. Fundamentally, it is a linear map from \mathbb{C}^n to \mathbb{C}^n .

To motivate the definition of the discrete Fourier transform, we’ll show that it can be derived—up to a scaling factor—by considering a Riemann sum approximation (with the trapezoidal rule) of a function’s Fourier coefficients. Since the DFT is a map from $\mathbb{C}^n \rightarrow \mathbb{C}^n$, we’ll work with the complex exponential form of the Fourier series, where the coefficients are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} dx.$$

First, recall the definition of the trapezoidal Riemman sum.

Definition 3.1 ([Trapezoidal approximation to integral](#)). Let g be an integrable function on some finite interval $[a, b]$. Then the trapezoidal Riemman sum approximation to $\int_a^b g(x)dx$ using $N + 1$ points is

$$(8) \quad I_h[g; a, b] := \frac{h}{2} \left(g(a) + \sum_{n=1}^{N-1} g(x_n) + g(b) \right)$$

where $h = (b - a)/N$ is the ‘grid-spacing’, $x_n = nh + a$ for $n = 0, 1, \dots, N$, and $x_0 = a$ and $x_N = b$.

Note that $I_n[g; a, b] \rightarrow \int_a^b g(x)dx$ as $h \rightarrow 0$, or equivalently, as $N \rightarrow \infty$.

Remark 3.2. The quantity (8) is usually called the *composite trapezoidal rule* in numerical analysis.

Exercise 3.3 ([Motivation for definition of discrete Fourier transform](#)). (a) Suppose some function g is L -periodic, i.e. $g(x + L) = g(x)$ for all $x \in \mathbb{R}$. Then true or false:

$$\int_0^L g(t)dt = \int_s^{s+L} g(t)dt, \quad \forall s \in \mathbb{R}?$$

(b) Let $f(x) = f(x + 2\pi) \forall x \in \mathbb{R}$, and let $f \in L^2([-\pi, \pi])$. Using part (a), write the complex Fourier coefficient c_k as

$$c_k = \frac{1}{\pi} \int_0^{2\pi} f(x)e^{-ikx} dx.$$

(c) Let $N > 0$, and let $h = 2\pi/N$. Show that the trapezoidal approximation

$$I_h[f(x)e^{-ikx}; 0, 2\pi]$$

(defined by (8)) to c_k equals

$$\frac{2\pi}{N} \sum_{n=0}^{N-1} f(x_n)e^{-2\pi i kn/N}.$$

Definition 3.4 (Discrete Fourier transform (DFT)). Let $y \in \mathbb{C}^n$, and label the components of y as $[y_0, y_1, y_2, \dots, y_{N-1}]^T$. The discrete Fourier transform of y is defined to be the vector $c \in \mathbb{C}^n$ with components

$$c_k = \sum_{n=0}^{N-1} y_n e^{-2\pi i kn/N}, \quad k = 0, 1, \dots, N-1.$$

The DFT then is a linear map $M : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with entries

$$(M)_{kn} = e^{-2\pi i kn/N}, \quad 0 \leq k, n \leq N-1.$$

Remark 3.5. Notice the constant $2\pi/N$ from Exercise 3.3(c) is missing from this definition. In general, the normalization is *not unique*, it's merely a matter of convention. The definition given here follows the convention of Matlab, numpy, and others.

Proposition 3.6 (Inverse DFT). Let $A \in \mathbb{C}^{n \times n}$ with

$$(A)_{kn} = \frac{1}{N} e^{2\pi i kn/N}, \quad 0 \leq k, n \leq N-1$$

so that for $c \in \mathbb{C}^n$, the n -th component of Ac is

$$(Ac)_n = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{2\pi i kn/N}.$$

Then $AM = MA = I$, where I is the identity map, i.e. A is the inverse discrete Fourier transform

We won't prove the result, but notice the inverse map simply has a positive sign in the complex exponential and a normalization factor of $1/N$.

3.1. Fast Fourier Transform: [one of “most important numerical algorithm of our time” (see IEEE's list)] One reason the DFT is so useful is that it can be computed quite rapidly. Computing the DFT amounts to performing a matrix-vector multiplication; for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, in general this operation is performed with $\sim n^2$ arithmetic operations—to compute one entry requires $\sim n$ operations, and there are n entries to compute.

The matrix M representing the DFT has some inherent symmetry, however, that enables the matrix-vector multiplication to be computed with only $\sim n \log n$ arithmetic operations (the inverse DFT shares this property as well).

The symmetry allows for a recursive-style algorithm that is responsible for the dramatic speed-up that's worth further exploration.

3.2. Additional exercises.

Exercise 3.7 (Conjugate symmetry for real input). Suppose that now that $y \in \mathbb{R}^n$ (obviously $y \in \mathbb{C}^n$ is still true, but all of the complex parts of the entries $y_n = 0$). Assume that N is even.

Show that for $1 \leq k \leq N/2$

$$\overline{c_k} = c_{N-k},$$

where $\bar{\cdot}$ denotes the complex conjugate.

Hint: consider multiplying by $1 = e^{-2\pi i N/N}$.

Exercise 3.8 (Computational exercise: data smoothing). In “Part 2: Summary of main ideas” Exercise 1.4, we found that the *smoothness*, or differentiability, of a function is related to the decay of its Fourier coefficients.

The same is true of a discrete signal; we'll examine a simple way to ‘clean’ noisy data here.

Download the python program and simple .txt files

`dft-smoothing.py` `dow.txt`

The latter contains the daily closing values of the Dow Jones Industrial Average, which is a measure of the average prices of the US stock market, from late 2006 until the end of 2010 (notice a precipitous drop sometime around 2008...).

The python script loads the Dow Jones data, takes its discrete Fourier transform (via the FFT, utilizing conjugate symmetry), sets some of the discrete Fourier coefficients to zero, then takes the inverse transform.

Experiment with different values of the parameter ‘`filter_ind`’. What is happening as you set more and more coefficients equal to zero?

Although more sophisticated techniques exist, the main idea is simply to ‘dampen’ or ‘mollify’ the effect of high-frequency components.

4. PART 4: POINTWISE COVERAGE OF FOURIER SERIES & GIBBS PHENOMENON

Given a square-integrable $f \in L^2([-\pi, \pi])$, we've shown that the trigonometric polynomials

$$T_n(x) = \frac{1}{2} \frac{1}{\pi} \langle f, 1 \rangle + \sum_{k=1}^n \frac{1}{\pi} (\langle f(x), \cos(kx) \rangle \cos(kx) + \langle f(x), \sin(kx) \rangle \sin(kx))$$

converge in $L^2([-\pi, \pi])$ to f . However, this does NOT imply that the Fourier series will converge *pointwise*; in other words, it may *not* be true that

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (\langle f(x), \cos(nx) \rangle \cos(nx) + \langle f(x), \sin(nx) \rangle \sin(nx))$$

for $x \in [-\pi, \pi]$. In general this is true of L^2 and pointwise convergence, and it's a useful thing to keep in mind in real analysis, as this exercise illustrates.

Exercise 4.1 (Relationship between L^2 and pointwise convergence). (a) Let $\{f_n\}$ be a sequence of continuous functions on the finite interval $[a, b]$ that converges uniformly to f , so that

$$\lim_{n \rightarrow \infty} \sup_{x \in [a, b]} |f_n(x) - f(x)| = 0.$$

Show that this implies convergence in L^2 :

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^2([a, b])} = 0.$$

(b) Construct a counterexample to the reverse statement. That is, can you think of a sequence of continuous functions $\{f_n\}$ converging in $L^2([0, 1])$ to f that does *not* converge pointwise uniformly?

So, when *does* a Fourier series converge pointwise? It could help to get some intuition by looking closely at a few examples.

Exercise 4.2 (Fourier series of smooth function). Let $f(x) = x^2$. Observe that $f \in L^2([-\pi, \pi])$ and $f(-\pi) = f(\pi)$.

(a) Compute the Fourier coefficients $\{a_0, a_1, b_1, a_2, b_2, \dots\}$ of f .

Hint: notice f is an even function, i.e. $f(x) = f(-x) \forall x$. What then can we immediately say about the coefficients $\{b_1, b_2, \dots\}$?

(b) Download the python program

`fourier-series-xsqr.py`

and enter your answer from part (a) where indicated. Run the program and observe how the partial sums T_n compare for various values of n . How close are the approximations?

Exercise 4.3 (Fourier series of square wave: Gibbs phenomenon). Let

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Observe that $f \in L^2([-\pi, \pi])$, but f of course is not continuous.

(a) Compute the Fourier coefficients $\{a_0, a_1, b_1, a_2, b_2, \dots\}$ of f .

Hint: notice f is an odd function, i.e. $f(x) = -f(-x) \forall x \neq 0$. What then can we immediately say about the coefficients $\{a_0, a_1, a_2, \dots\}$?

(b) Download the python program

`fourier-series-squarewave.py`

and enter your answer from part (a) where indicated. Run the program and observe how the partial sums T_n compare for various values of n . How close are the approximations? What you observe is commonly known as the *Gibbs (or Gibbs-Wilbraham) phenomenon*.

The above examples suggest that if a function is discontinuous at some point $x^* \in [-\pi, \pi]$, there may be a lack of point-wise convergence there. So, what are sufficient conditions to ensure point-wise convergence? We will next show that *Lipshitz continuity* is sufficient to guarantee pointwise convergence.

Definition 4.4 (Dirichlet kernel). The function

$$D_n(x) = \frac{\sin((n+1/2)x)}{2\sin(x/2)}$$

is called the Dirichlet kernel. Observe that it is an even function: $D_n(x) = D_n(-x)$.

Exercise 4.5 (Dirichlet kernel identity). Show that

$$D_n(x) = \frac{\sin((n+1/2)x)}{2\sin(x/2)} = \frac{1}{2} + \sum_{k=1}^n \cos(kx).$$

This means that the Dirichlet kernel is simply a n -th degree trigonometric polynomial with coefficients $a_0 = a_1 = \dots = a_n = 1$ and $b_k = 0$ for each k .

Hint: Multiply the sum on the right-hand side by $2\sin(x/2)$ and use the identity

$$2\sin(x/2)\cos(kx) = \sin((k+1/2)x) - \sin((k-1/2)x);$$

the resulting sum should telescope.

Exercise 4.6 (Dirichlet formula for n -th partial Fourier sum). Let

$$a_k = \frac{1}{\pi} \langle f(x), \cos(kx) \rangle, \quad b_k = \frac{1}{\pi} \langle f(x), \sin(kx) \rangle, \quad k = 0, 1, \dots, n$$

be the Fourier coefficients of $f \in L^2([-\pi, \pi])$ (where again, $b_0 = 0$), and let $f(-\pi) = f(\pi)$. Let $T_n(x) = a_0/2 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$.

(a) Use the definition of the Fourier coefficients to write T_n as

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^n (\cos(kx)\cos(kt) + \sin(kx)\sin(kt)) \right] f(t) dt$$

(b) Use the trigonometric identity

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

to write T_n as

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x-t)f(t) dt$$

(c) For any integrable function g that is L periodic (so that $g(t) = g(t+L) \forall t \in \mathbb{R}$), convince yourself that $\int_0^L g(t)dt = \int_s^{s+L} g(t)dt$ for any $s \in \mathbb{R}$.

(d) Use the fact that D_n is an even function, that both D_n and f are 2π -periodic, and your observation from part (c) to show that

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(s)f(x+s) ds.$$

This is known as the *Dirichlet formula* for the n -th partial Fourier sum.

We now have the ingredients to prove the pointwise convergence theorem.

Exercise 4.7 (Pointwise convergence of Fourier series for Lipschitz f). Let $f \in L^2([-\pi, \pi])$ and let T_n be the n -th partial sum of its Fourier series. Let f be extended to all of \mathbb{R} by the periodicity condition $f(x + 2\pi) = f(x)$. Suppose f satisfies a *Lipschitz condition* at x , meaning

$$|f(x+t) - f(x)| \leq C|t|, \quad |t| < \delta$$

for some positive constants C and δ . We'll show T_n converges *pointwise* to f at x : $\lim_{n \rightarrow \infty} T_n(x) = f(x)$.

(a) Use the Dirichlet formula from Exercise 4.6(d) to write the pointwise difference $T_n(x) - f(x)$ as

$$T_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi_x(t) \sin\left(\left(n + \frac{1}{2}\right)t\right) dt$$

where

$$\varphi_x(t) = \frac{f(x+t) - f(x)}{2 \sin(t/2)}.$$

(b) Show that $\varphi_x \in L^2([-\pi, \pi])$. To do this, break the region of integration $[-\pi, \pi]$ into two pieces: where $|t| \leq \delta$, and everywhere else (remember, we don't know δ , but, if $\delta > \pi$ were true, then we'd really only have one piece). Where $|t| \leq \delta$, show

$$|\varphi_x(t)| \leq C,$$

where C is the Lipschitz constant of f . Note of course that $C \in L^2([-\pi, \pi])$. Where $|t| > \delta$, show the denominator of $\varphi_x(t)$ is bounded below by 0, and use that $f \in L^2([-\pi, \pi])$.

(c) Use the trig identity

$$\sin\left(\left(n + \frac{1}{2}\right)t\right) = \cos(t/2) \sin(nt) + \sin(t/2) \cos(nt)$$

to write $T_n(x) - f(x)$ as

$$(9) \quad \frac{1}{\pi} \langle \varphi_x(t) \cos(t/2), \sin(nt) \rangle + \frac{1}{\pi} \langle \varphi_x(t) \sin(t/2), \cos(nt) \rangle.$$

(d) In (b) we showed $\varphi_x \in L^2$. Are $\varphi_x(t) \cos(t/2)$ and $\varphi_x(t) \sin(t/2)$ also square-integrable on $[-\pi, \pi]$? Based on your answer, what happens to (9) as $n \rightarrow \infty$? Conclude then that

$$\lim_{n \rightarrow \infty} |T_n(x) - f(x)| = 0$$

as desired.

5. PART 5: FOURIER TRANSFORM ON THE REAL LINE \mathbb{R}

So far we've developed theory and intuition for two types of Fourier maps.

The first is typically termed the 'Fourier series' of a function that is defined on the torus—typically the periodic interval $[-\pi, \pi]$. As we saw, ultimately the Fourier series of a function is a map $\mathcal{F} : L^2([-\pi, \pi]) \rightarrow l^2(\mathbb{Z})$, that is, a map from the space of square integrable functions on a closed and bounded interval to square-summable sequences.

Another map we've discussed is the Discrete Fourier Transform (DFT). Although we motivated the DFT by looking at a trapezoidal approximation to the Fourier coefficients of some $f \in L^2([-\pi, \pi])$, at the end of the day it's simply a linear map M (i.e. a matrix) on the space of n -dimensional complex vectors: $M : \mathbb{C}^n \rightarrow \mathbb{C}^n$.

There is another Fourier transform that is defined for functions that are integrable on the real line \mathbb{R} which we'll now study in some detail.

Definition 5.1 (Notation). Note that the following two notations are equivalent:

$$\int_{-\infty}^{\infty} \leftrightarrow \int_{\mathbb{R}}$$

Definition 5.2 (L^p spaces). Let $p > 1$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$. If

$$\int_{\mathbb{R}} |f(x)|^p dx < \infty,$$

then we say $f \in L^p(\mathbb{R})$.

Note that if $p = 2$, this is the inner-product space L^2 that we've already defined above (noting, however, that in the previous case, we only considered functions on $[-\pi, \pi]$, not all of the real line).

Definition 5.3 (Fourier transform). Let $f \in L^1(\mathbb{R})$ so that

$$\int_{\mathbb{R}} |f(x)| dx < \infty.$$

Then define the *Fourier transform* of f to be

$$\widehat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx} dx.$$

Note that by assumption,

$$|\widehat{f}(k)| \leq \int_{\mathbb{R}} |f(x)e^{-ikx}| dx < \infty \quad \forall k \in \mathbb{R},$$

since $f \in L^1(\mathbb{R})$.

Exercise 5.4 (Fourier transform of characteristic function). Let $R > 0$ be some constant, and define

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq R \\ 0 & \text{for } |x| > R \end{cases}$$

as the characteristic function on the set $[-R, R]$.

(a) Compute the Fourier transform \widehat{f}

(b) Is $\widehat{f} \in C(\mathbb{R})$? What is the value of $\lim_{k \rightarrow \pm\infty} \widehat{f}(k)$? Is $\widehat{f} \in L^1(\mathbb{R})$?

Exercise 5.5 (Basic properties of the Fourier transform). (a) True or false: the Fourier transform $\widehat{\cdot}$ is a linear operator?

(b) Show that translation of a function maps to a complex rotation under $\widehat{\cdot}$; that is, show that if $g(x) = f(x - b)$, then

$$\widehat{g}(k) = e^{-ibk} \widehat{f}(k), \quad \forall b \in \mathbb{R}.$$

(c) Show that rescaling a function by a maps to rescaling by $1/a$ and an additional normalization; that is, show that if $g(x) = f(ax)$, then

$$\widehat{g}(k) = \frac{1}{|a|} \widehat{f}(k/a), \quad \forall a \in \mathbb{R}, a \neq 0.$$

Exercise 5.6 (Continuity and decay at infinity). Let $f \in L^1(\mathbb{R})$ and let $\widehat{f}(k)$ be its Fourier transform.

(a) Show that $\widehat{f}(k)$ is uniformly continuous on \mathbb{R} .

(b) Use your answer to Exercises 5.4(b) and 5.5 to argue that functions of the form

$$(10) \quad s_n(x) = \sum_{k=1}^n c_k \chi_{I_k}$$

satisfy $\widehat{s}_n(k) \rightarrow 0$ as $k \rightarrow \pm\infty$, where each $c_k \in \mathbb{R}$ and each χ_{I_k} are so-called characteristic functions on the closed and bounded intervals $I_k = [a_k, b_k]$, meaning

$$\chi_{I_k}(x) = \begin{cases} 1 & \text{for } x \in I_k \\ 0 & \text{for } x \notin I_k. \end{cases}$$

(c) A quite useful fact from real analysis is that functions of the form (10), termed *simple functions*, are *dense* in $L^1(\mathbb{R})$. More specifically, this means that for any $f \in L^1(\mathbb{R})$, $\exists \{s_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f(x) - s_n(x)| dx = 0.$$

Use this density property to show that for any $f \in L^1(\mathbb{R})$, $\widehat{f}(k) \rightarrow 0$ as $k \rightarrow \pm\infty$.

Exercise 5.7 (Differentiability and Convolution). (a) Let $g(x) = xf(x)$ and assume $g \in L^1(\mathbb{R})$.

(b) Let $f \in C^1(\mathbb{R})$ and let both $f, f' \in L^1(\mathbb{R})$. Then show

$$\widehat{f'}(k) = ik \widehat{f}.$$

(c) Suppose that $f, g \in L^1(\mathbb{R})$, and define the *convolution* $f * g$ to be

$$(f * g)(x) = \int_{\mathbb{R}} f(s)g(x - s) ds.$$

Use Fubini's theorem to show that

$$\widehat{f * g}(k) = \widehat{f}(k) \widehat{g}(k).$$

Exercise 5.8 (Fourier transform of a Gaussian). Let

$$f(x) = e^{-x^2/2}$$

be a Gaussian distribution, a function which plays an important role in probability theory and many other fields. We'll show that f is an *eigenfunction* of the Fourier transform \widehat{f} .

(a) Use the differentiability property shown in parts (a) and (b) of Exercise 5.7 to show \widehat{f} satisfies the ordinary differential equation (ODE)

$$k \widehat{f}(k) + \frac{d}{dk} \widehat{f}(k) = 0.$$

(b) Compute the constant

$$C := \widehat{f}(0) = \int_{\mathbb{R}} e^{-x^2/2} dx$$

(c) Solve the ODE from (a) using the constant C from (b) to obtain an expression for $\widehat{f}(k)$. What is the eigenvalue of $\widehat{\cdot}$ associated with the Gaussian f ?

(d) Finally, using Exercise 5.5(c), compute the Fourier transform of $g(x) = e^{-ax^2/2}$

Exercise 5.9 (Poisson Summation Formula). Let $f \in L^1(\mathbb{R})$ and $f \in C(\mathbb{R})$, and assume

$$g(t) = \sum_{k \in \mathbb{Z}} f(t+k)$$

converges uniformly on any bounded interval. Further assume that

$$\sum_{n \in \mathbb{Z}} \widehat{f}(2\pi n)$$

is absolutely convergent.

(a) True or false: $g(t)$ is both continuous and periodic? If it's periodic, what is the period?

(b) Let $h(x) = h(x+2\pi) \forall x \in \mathbb{R}$ and assume $h \in L^2([-\pi, \pi])$. Show with a change of variables that the n -th Fourier coefficient (in complex form) of h can be written as

$$c_n = \int_0^1 h(s) e^{-2\pi i n s} ds.$$

(c) Use parts (a) and (b) and the definition of the Fourier transform of an L^1 function to show that the n -th Fourier coefficient c_n of $g(t) = \sum_{k \in \mathbb{Z}} f(t+k)$ satisfies

$$c_n = \widehat{f}(2\pi n).$$

(d) The hypothesis that $\sum_{n \in \mathbb{Z}} |\widehat{f}(2\pi n)| < \infty$ implies the series

$$\sum_{n \in \mathbb{Z}} \widehat{f}(2\pi n) e^{2\pi n t}$$

converges uniformly in t

5.1. **Additional Exercises.** First, we give the definition of the inverse Fourier transform.

Definition 5.10 (Inverse Fourier Transform). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be infinitely differentiable with compact support, meaning the set of values for which $f(x) \neq 0$ is contained in a closed and bounded interval. Then

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(k) e^{ikx} dk$$

where \widehat{f} is the Fourier transform of f .

Remark 5.11. Describing the general conditions necessary to ensure the inverse Fourier transform is well defined takes some time to develop and we won't pursue it here. In particular, merely ensuring $f \in L^1(\mathbb{R})$ is *not* enough to ensure that the inverse transform exists. However, whenever $\widehat{f} \in L^1(\mathbb{R})$, clearly the inverse transform is well defined. A sufficient condition for this to happen is for f to be smooth and compactly supported.

Exercise 5.12 (Infinite product representation of sine function). Consider $f(x) = \cos(cx)$ defined for $x \in [-\pi, \pi]$, and assume $c \notin \mathbb{Z}$. Although f does not have the period 2π , it is an even function, so that $f(-\pi) = f(\pi)$. Thus, it can be periodically extended to all of \mathbb{R} .

(a) Convince yourself that the periodically extended function satisfies a Lipschitz condition $\forall x \in \mathbb{R}$, and hence that it has a pointwise convergent Fourier series.

(b) Show that the Fourier coefficients a_n of $f(x)$ are

$$a_0 = \frac{2}{c\pi} \sin(c\pi), \quad a_n = \frac{2c}{\pi} \sin(c\pi) \frac{(-1)^n}{c^2 - n^2}, \quad n = 1, 2, 3, \dots$$

(Why are all the $b_n = 0$?)

Hint: the trigonometric identity $2 \cos(cx) \cos(nx) = \cos((c+n)x) + \cos((c-n)x)$ will be useful.

(c) Using the Fourier series just computed for $\cos(cx)$ and your answer from part (a), evaluate both $\cos(cx)$ and its Fourier series at $x = \pi$. We'll now consider c as the variable in the problem, keeping in mind that c is assumed not to be an integer. Change notation from c to t to obtain

$$(11) \quad \pi \cot(\pi t) - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}, \quad t \notin \mathbb{Z}.$$

(d) Assume $0 < \epsilon < x < 1$ and integrate both sides of (11) term-by-term from $t = \epsilon$ to $t = x$. (Why is this justified?) Take the limit as $\epsilon \rightarrow 0$, and then exponentiate both sides to derive the infinite product representation

$$(12) \quad \sin(\pi x) = \pi x (1 - x^2) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \dots$$

(e) Evaluate (12) at $x = 1/2$ to obtain the *Wallis product formula*:

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{4^2}\right) \left(1 - \frac{1}{6^2}\right) \dots$$

Exercise 5.13 (Solution of diffusion equation (a.k.a. heat equation)). Consider the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2} = 0$$

posed on the real line \mathbb{R} with initial condition $u(x, t = 0) = f(x)$ for some smooth function f with compact support.

(a) Take the Fourier transform of both sides of the PDE to obtain an ordinary differential equation (ODE) with initial condition $\widehat{u}(k, t = 0) = \widehat{f}(k)$, and then solve the ODE.

Note: you may assume that the time-derivative $\partial/\partial t$ and the Fourier transform $\widehat{\cdot}$ commute.

(b) Use a method similar to that outlined in Exercise 5.8 to compute the *inverse Fourier transform* of $e^{-k^2 t}$. This function is called the *heat kernel* on \mathbb{R} .

(c) Use the convolution property derived in Exercise 5.7(c) to solve the PDE for $u(x, t)$.

Exercise 5.14 (Application of Poisson summation). Use the Poisson summation formula to calculate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + t^2} = \frac{\pi}{t} \frac{1 + e^{-2\pi t}}{1 - e^{-2\pi t}}, \quad t > 0.$$

Use this result to show that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Hint: Subtract $1/t^2$ from both sides of the equation and find the limit as $t \rightarrow 0$. Utilize the Taylor the expansion of the exponential function.

Exercise 5.15 (Wirtinger's inequality). Suppose $f \in C^1([0, 2\pi])$ and that $f(0) = f(2\pi)$. Additionally assume $\int_0^{2\pi} f(x) dx = 0$. Use Parseval's relation to show

$$\int_0^{2\pi} (f(x))^2 dx \leq \int_0^{2\pi} (f'(x))^2 dx$$

where the inequality is *strict* unless $f(x) = A \cos(x) + B \sin(x)$ for some constants A and B .

Exercise 5.16 (Isoperimetric inequality). Complete exercise [24] on pages 242–243 in Duren's text to show that among all simple closed curves of a fixed length, the circle encloses the region with the largest area.