# Contents

1 Introduction .......................... 3

2 Why you need DA (borrowed shamelessly from some website) .......................... 4

3 Basics .................................. 7
   3.1 What is dimensionality? .................. 7
   3.2 Dimensionless Things ................... 9
   3.3 Dimensional Classes .................... 10
   3.4 Examples of dimensionality ............ 11
   3.5 The Canonical Class .................... 12
   3.6 Dimensional Homogeneity ............... 12
   3.7 Using DA to Check Solutions ........... 13

4 More Fundamentals ..................... 15
   4.1 Dimensional dependence ................ 15
   4.2 Basis ................................. 16
   4.3 Testing A Possible Basis ............... 17
   4.4 Analogy to Vectors and Vector Spaces .. 18
   4.5 Dimensionless Forms ................... 18
   4.6 Examples ............................... 19

5 Buckingham $\Pi$ Theorem ............... 20
   5.1 An example ............................. 20
   5.2 Statement of the Theorem ............... 20
   5.3 A Nice, Simple Recipe for Applying the Theorem .. 21

6 Examples of the Buckingham $\Pi$ Theorem .......................... 22
   6.1 Relating Basic Quantities ............... 22
   6.2 Simplifying Governing Equations ....... 24
   6.3 Data Collapse .......................... 25

7 This is great stuff! What could possibly go wrong? .................. 26
1 Introduction

Dimensional analysis (DA) is an extremely useful tool for the physicist, and it is used implicitly in virtually all advanced physics courses. However, even though it is used frequently, it is not usually taught properly or as a subject in its own right. Or if it is taught, it often is not taught thoroughly and in a way that makes it rigorous and useful. Part of the difficulty may be a sense of caution that should be developed along with its introduction, because the careless use of dimensional analysis can also lead to amazingly dopey errors.

In this short introduction, we will attempt to give a sufficiently rigorous treatment. At the same time, we will give several examples of how to use it correctly, and also some examples to demonstrate possible pitfalls.
2 Why you need DA (borrowed shamelessly from some website)

*Dimensional Analysis is not for everyone, but it is probably for you!*

First of all, who should avoid Dimensional Analysis?

**TOP 4 REASONS FOR NOT USING DIMENSIONAL ANALYSIS:**

1. Let’s say you’re super-intelligent and enjoy solving relatively simple problems in the most complex manner.

2. Let’s say you’re tired of always getting the correct answers.

3. Let’s say you’re an arty type and you can’t be confined by the structure of DA. You like messy solutions scribbled all over the page in every which direction. It’s not that you want to make a mistake. But you really don’t care that much about the answer. You just like the abstract design created by the free-wheeling solution... and the freedom from being confined by structure.

4. Let’s say that you have no interest in going out on a date or graduating, and you don’t mind being unpopular, unattractive, ignorant, insecure, uninformed, and unpleasant.

Otherwise, *you need dimensional analysis!*

**Testimonials:**

"I was studying for my Ph.D. in Physics at Clemson. I dozed off while Prof. Daw taught us Dimensional Analysis. I never quite got the hang of it. It irritated me... all of those matrices and determinants. I never really liked determinants. Although my grades had been pretty high, and I was doing pretty well on my research project, I got a "C" in Graduate Quantum and had to drop out of grad school. It was not long before I started on drugs, and then crime to support my drug habit. Recently I learned Dimensional Analysis and realize how simply it could have solved all of my problems. Alas, it is too late. I won’t get out of prison until 2012 and even then, my self image is permanently damaged. I attribute all of my problems to my unwillingness to learn Dimensional Analysis." — Jane
"I thought I knew everything and that lab work were the only thing that mattered for my PhD. When Prof. Daw taught our class Dimensional Analysis, I didn’t care about it at all. I was making plans for the weekend with my girlfriend who loved me because I was an experimentalist and not because of my analytical skills. While other students were solving problems using Dimensional Analysis, I was calibrating my thermocouple. Then one day I was hit hard. My advisor said I had to understand what was going on in my experiment. I was despondent. My girlfriend deserted me. My parents, who used to brag about having a physicist in the family, started getting on my case about graduating. I decided to throw myself into my school work. But I couldn’t understand anything. I would get wrong answers all of the time. I now realize that my failure came from never having learned Dimensional Analysis. Alas, I thought everyone else was smarter. After the constant humiliation of failing I finally gave up. I am worthless. I have no friends, no skills, no interests. I have now learned Dimensional Analysis, but it is too late.” — Bill

“I was at home, sick with the flu when Prof. Daw taught my class about Dimensional Analysis. Despite opportunities given to me to make up the assignments that I had missed, I chose to not do them. I thought that my mathematical abilities were already sufficient. How wrong I was! It’s been five years since I took that class. Now I spend my afternoons panhandling at traffic lights, hoping for passersby to give me spare change. If I’m lucky enough to scam a buck after a day’s work, I’m still not sure if my hourly rate makes cents.” — Mario

**Hard Data:**

The evidence: Studies at Clemson from 1998-2004 show that 100% of first year grad students who do not use and understand Dimensional Analysis are seriously insecure by the time they must defend their thesis. Damage done from this deprivation is probably permanent and cannot be overcome by learning the method later in life. We recommend mastering this skill your first year.

83% of the PhD students who graduated from 1994-99 admitted that they enjoyed solving problems with Dimensional Analysis in order to impress and confuse their colleagues. Of the remaining 17%, 11% never had a date on
the weekend and 6% could not keep remember to zip their fly.

Convinced yet? Very well! Let me teach you some of the basics first.
3 Basics

3.1 What is dimensionality?

Physical quantities have dimensionality, by which we mean they have particular types such as length or time. As such, they are measured by comparison to standards, and the numerical value associated with a physical quantity has meaning only with respect to that standard. That is, to say that a car has length 5 is meaningless unless you give the units.

The most basic physical quantity is length, because it reflects the extension of material bodies. If we change the units, then the measure of a length will change even though the property of the length itself remains unchanged. (That is, the numerical value associated with the measurement of the physical property changes, but the physical thing itself is unchanged by our choice of units for the measurement.) All lengths measured against one standard will change numerical value in the same proportional way if measured against another standard.

Length as a property has a nature which is independent of the choice of units. That is, we know what “length” means as a property, independent of how it is measured. It is a length whether it is measured in meters or inches. In that same way, all lengths are similar to each other, because they are the same type of property.

Time is also basic because it is a measure of change, which is also a fundamental characteristic of material being. Time has a nature independent of the choice of units. All times are similar to each other, because they are of the same type.

To reflect the distinction, we say that $x$ and $t$ are measures of a position and a time, and as such are fundamentally different. We understand that position and time have different natures, even though we represent the measures of both things by numbers. Some of that nature is captured by the idea that meters and seconds are fundamentally different. The difference is much deeper, but the distinction of units of length from units of time captures part of the difference.

To express this understanding in mathematical form, we adopt a notation which originated with Maxwell. The “dimensionality” of a physical variable tells us something about the nature of the property that it is a measure of. So, “lengthness” is denoted by $L$, and “timeness” is denoted by $T$, and the
dimensionality of a variable is expressed as a square bracket. So, if \( x \) is a position and \( t \) is a time, then

\[
[x] = L \quad \text{and} \quad [t] = T
\]  

(1)

Some quantities — take speed, for example — are combinations of more fundamental things. We are familiar with how this goes, because it’s no surprise to write of a speed \( v \) that its dimensionality is given by

\[
[v] = L/T = LT^{-1}
\]  

(2)

Momentum is a property distinct from position and time. Because of its conceptual and historical significance, some people have proposed that momentum should be given its own unit of measure, called the “Buridan” after the 14th century scientist (Jean Buridan), who was the first to express the idea of momentum clearly. So, if \( p \) is the measure of momentum, then

\[
[p] = P
\]  

(3)

where \( P \) would note the type of the property of momentum.

To describe a great many examples of motion, we need a combination of these three types of properties: length, time, and momentum. For example, the motion of a mass on a spring can be described by those three basic types. The dimensionality of all physical quantities involved in the motion of a mass on a spring can be expressed in terms of \( L \), \( T \), and \( P \).

For example, the governing equation (the equation which governs the dynamics of the system) for a mass on harmonic spring is simply:

\[
\frac{dp(t)}{dt} = -kx(t)
\]  

(4)

where we recognize that \( p \) and \( x \) are dynamical variables — that is, dependent on time and we also see that there is a spring constant \( k \) which represents the stiffness of the spring. The dimensionality of the left-hand-side (LHS) is expressed:

\[
\left[ \frac{dp(t)}{dt} \right] = \left[ \frac{p}{t} \right] = PT^{-1}
\]  

(5)

(note that the derivative is just a ratio, and so the dimensionality comes from that). The right-hand-side (RHS) has dimensionality:

\[
[-kx(t)] = [k][x] = [k]L
\]  

(6)

(notice that the sign has vanished: the dimensionality of a quantity is independent of sign). This allows us to express the dimensionality of the spring
constant $k$ as $[k] = PT^{-1}L^{-1}$.

I’ll mention something which might be obvious, but just to be complete, notice that

$$[p^a] = [p]^a$$

for any property $p$.

We will see presently that it is possible to express the dimensionality of all physical quantities in this system completely by using the three basic dimensionalities $L - T - P$.

### 3.2 Dimensionless Things

Note that a numerical constant is dimensionless. So $\pi$ or 1/2 as numerical constants devoid of units would have dimensionality 1, as in:

$$[\pi] = 1$$

$$[1/2] = 1$$

Finally, angles are dimensionless, as can be demonstrated by remembering that an angle is a ratio of an arc length to a radius, both of which have dimensionality $L$. Thus, any angle $\theta$

$$[\theta] = 1$$

It is best always to express angles in radians. Do not be confused with the fact that angles are sometimes expressed as degrees — radians and degrees are not a unit in the strict sense of the term.

Unit vectors are dimensionless, as might be suggested by their name. For example, consider describing a position vector $\vec{r}$ in terms of Cartesian components $x, y,$ and $z$, and the unit vectors $\hat{x}, \hat{y}, \hat{z}$. Then we have

$$\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$$

Considering that $[\vec{r}] = L$, it is easy to see that the unit vectors must be dimensionless.

Finally, arguments of inhomogeneous functions must be dimensionless. That applies to functions such as exp, ln, cos, etc. So if one has an expression $\cos (\omega t)$, then the product of $\omega$ and $t$ must be dimensionless, or else the expression will not be physically meaningful. One can see this from doing a Taylor’s series of the function, and then requiring that all terms have the same dimensionality.
3.3 Dimensional Classes

In the previous section, we considered a simple example of dynamics whose dimensionalities could be expressed completely in terms of $L$, $T$, and $P$. Generally, we speak of this as an example in the $L - T - P$ class. We consider now other classes.

Consider mass, which is distinct from length, time, and momentum, but it can be expressed in terms of those properties. The fundamental definition of mass is in terms of the speed that a body has as a result of the momentum it possesses. In mathematical expression, the measure of mass is given by

$$[m] = \frac{[P]}{[v]} = \frac{[p]}{[v]} = PTL^{-1}$$

(12)

While momentum is fundamental and mass is conceptually a derived property, it may be mathematically convenient to adopt a different choice. That is, we could chose to express the dimensionality of mass as

$$[m] = M$$

(13)

and then treat the momentum as defined in terms of $M$, $L$, and $T$, so that

$$[p] = MLT^{-1}$$

(14)

in which case we are working in the $L - T - M$ class.

Or, we could work in the $T - M - P$ class, and so forth. It is clear that there are only three independent dimensions among $P$, $L$, $M$, and $T$, and so we can only choose three to be basic in some sense. We will explore this in greater detail in what follows. For now, let us note that we have some freedom of choice in the selection of the class, and as always we make the choice based on what will either give more insight into the solution or make the solution simpler.

Sometimes it may be convenient to consider the $E - L - T$ class, where $E$ is an energy. In that case, the dimensionality of a mass $m$ or a momentum $p$ in the $E - L - T$ class would be:

$$[m] = ET^2L^{-2}$$

(15)

$$[p] = ETL^{-1}$$

(16)
3.4 Examples of dimensionality

Let’s work some examples of dimensionality.

Using the \( LTP \) class, find the dimensions of the following parameters:

- \( x \) (position or displacement)
- \( t \) (time)
- \( v \) (velocity)
- \( p \) (momentum)
- \( m \) (mass)
- \( F \) (force)
- \( E \) (energy)
- \( P \) (pressure)

Solution:

\[
\begin{align*}
[x] &= L \quad (17) \\
[t] &= T \quad (18) \\
[v] &= LT^{-1} \quad (19) \\
[p] &= P \quad (20) \\
[m] &= L^{-1}TP \quad (21) \\
[F] &= T^{-1}P \quad (22) \\
[E] &= LT^{-1}P \quad (23) \\
[P] &= L^{-2}T^{-1}P \quad (24)
\end{align*}
\]

We could alternatively have worked this in the \( LTM \) class. In which case, we would have this solution:

\[
\begin{align*}
[x] &= L \quad (25) \\
[t] &= T \quad (26) \\
[v] &= LT^{-1} \quad (27) \\
[p] &= MLT^{-1} \quad (28) \\
[m] &= M \quad (29) \\
[F] &= MLT^{-2}P \quad (30) \\
[E] &= MLT^{-2} \quad (31) \\
[P] &= ML^{-1}T^{-2} \quad (32)
\end{align*}
\]
3.5 The Canonical Class

Electric charge is another property of some material bodies. Dimensionally electric charge is distinct from position, length, momentum, mass. In problems involving electrodynamics, we must use then the dimensionality $Q$, from which the charge on a sphere $q$, for example, has dimensionality

$$[q] = Q$$

(33)

There are also other physical quantities, such as lepton number, baryon number, and so forth, which are different types of properties, and therefore have independent dimensionality. Problems involving those quantities would require augmenting the class. However, we will not consider those problems here.

For almost all of the problems encountered in the core courses of mechanics, dynamics, electricity and magnetism, and quantum mechanics can be treated with the $L - T - P - Q$ class, and we will designate that the canonical class. However, as we noted before, it is possible that another class will give more insight or a simpler solution. As you work through examples and become more familiar with DA, you will begin to see in advance how one class may be have an advantage over another. One might try to express general guidelines, but actually experience is the best guide.

For example, if there are no charges present, then obviously we can drop the $Q$ dimensionality as unnecessary.

Whatever class you choose, you should get in the habit of writing down the class explicitly when you begin your analysis. Write something like “Using the $LTM$ class”.

One more term. The “order” of the class it the number of basic types in the class. The order of $LTM$ is 3.

3.6 Dimensional Homogeneity

The fundamental theorem of dimensional analysis is quite simple:

**Theorem:** The dimension of any physical quantity is a power law monomial.

For example, in the $LMTQ$-class, any and all quantities must be expressed in this simple form:

$$[\text{quantity}] = M^a L^b T^c Q^d$$
where the exponents will be different for different quantities, and sometimes 0.

This fundamental theorem might be obvious to you if you’ve had courses in physics, but nonetheless it might surprise you to see it stated explicitly. It is a reflection of the nature of physical being, including the fact that a property is unchanged by a change in the way it is measured.

Here are a couple of useful corollaries:

*Corollary:* Both sides of a physically meaningful equation must have the same dimension.

If two things are equal, then they must scale in precisely the same way under a change of units, because otherwise equality in one system of units would be an inequality in another.

*Corollary:* All terms of a sum in a physically meaningful expression must have the same dimension.

If we have \( a + b \), then \([a] = [b]\) or else the sum is physically meaningless, because it will not itself conform to the Fundamental Theorem. This would be, literally, like mixing apples and oranges.

### 3.7 Using DA to Check Solutions

We have only begun to develop DA, but even what we have discovered so far can be very helpful. The Fundamental Theorem and its corollaries are useful for checking the manipulation of physics equations, if the quantities in the equations have dimensionality. All physics solutions must be dimensionally consistent, by which we mean that they conform to the requirements in the previous section.

Let’s work an example.

Determine if the following equation is dimensionally correct. If not, suggest a simple change that would make it correct (at least dimensionally correct).

\[ P = \sqrt{\rho gh} \], where \( P \) is pressure, \( \rho \) is mass density, \( g \) is gravitational acceleration, and \( h \) is height.
We begin by working out the dimensionality of all the parameters in the equation. Let’s use the $MLT$ class.

\[ [P] = [\text{force/area}] = ML^{-1}T^{-2} \quad (34) \]
\[ [\rho] = ML^{-3} \quad (35) \]
\[ [g] = LT^{-2} \quad (36) \]
\[ [h] = L \quad (37) \]

We can see then that the product $\rho gh$ has dimensionality $ML^{-1}T^{-2}$ but the square root of that (which is the RHS) has dimensionality $M^{1/2}L^{-1/2}T^{-1}$, which does not match the dimensionality of the LHS. It would be at least dimensionally correct if the square root were removed. (It might not be physically correct!)

An error in the dimensionality of a solution indicates an error in the derivation. One can then test steps in the derivation to narrow down where the error occurred.

This much is very useful, but there is much, much more! Read on!
4 More Fundamentals

4.1 Dimensional dependence

Have you ever seen a textbook suddenly introduce a system of units in which some of the parameters of a problem take a value equal to 1? For example, in some problems it is convenient to set $\hbar = c = 1$. Seems plausible, but how far can we take this? And what happens to the values of the other parameters? Stay tuned — you’ll not only learn how far you can go with this, but you can also find out how to do it yourself to any problem you come across.

First, we introduce some simple concepts: dimensional dependence and dimensional independence.

Two physical quantities which are of the same type — that is, which have the same dimensionality — are said to be dimensionally dependent. This is because if we change the system of units in such a way that one quantity changes, the other must scale by the same factor. In fact, note that the ratio of two such quantities would be dimensionless and therefore a constant independent of the system of units.

For example, take two distances $x$ and $y$, each with dimensionality $L$. The ratio $x/y$ is dimensionless. We say that the pair $x,y$ are dimensionally dependent.

This concept can be extended to include more than a pair of parameters. Consider three quantities: $x$, $t$, and $v$ (position, time, and velocity). The form $x/(vt)$ is dimensionless, and must therefore be independent of the choice of units. Therefore, this trio is dimensionally dependent.

Parameter $p$ is *dimensionally dependent* on parameter $q$ if there exists $a$ such that $[pq^a] = 1$.

Parameter $p$ is *dimensionally dependent* on parameters $q_1$, $q_2$ if there exists $a$, $b$ such that $[pq_1^a q_2^b] = 1$.

Parameter $p$ is *dimensionally dependent* on parameters $q_1$, $q_2$, $q_3$ if there exists $a$, $b$, $c$ such that $[pq_1^a q_2^b q_3^c] = 1$.

If parameters are not dimensionally dependent, then ... well, you get the idea.
Parameter $p$ is *dimensionally independent* of parameter $q$ if there exists no $a$ such that $[pq^a] = 1$.

Parameter $p$ is *dimensionally independent* of parameters $q_1$, $q_2$ if there does not exist $a$, $b$ such that $[pq_1^aq_2^b] = 1$.

Parameter $p$ is *dimensionally independent* of parameters $q_1$, $q_2$, $q_3$ if there does not exist $a$, $b$, $c$ such that $[pq_1^aq_2^bq_3^c] = 1$.

Let’s look at some examples. Consider the two parameters $x$ and $l$, both representing length. We see that $[x/l] = 1$, so they are dimensionally dependent. By contrast, consider $x$ and $m$, the latter being a mass. There is no value of $a$ which will make the quantity $xm^a$ dimensionless, so those two parameters are dimensionally independent. Finally, consider the set of three parameters: $\omega$ (a frequency), $g$ (gravitational acceleration), and $l$ (length). The combination $\omega \sqrt{l/g}$ is dimensionless, so these three are dimensionally dependent.

### 4.2 Basis

Now we consider that if we have a set of several parameters, some of which are dimensionally dependent on each other, we might be able to form a subset which are dimensionally independent among themselves and in terms of which the remaining parameters can be expressed.

Consider, for example, the trio from before: $x$, $t$, and $v$ in the $LT$ class. Because there are only two types of independent parameters (because the class is order 2), we could separate the list into two subsets. Let’s pick $x$ and $t$ to form the “basis”. We can express the dimensionality of remaining parameter ($v$) in terms of the dimensionality of $x$ and $t$, like this:

$$[v] = [x] / [t]$$

(38)

or

$$\left[ \frac{vt}{x} \right] = 1$$

(39)

What would constitute a basis? The parameters must be dimensionally independent of each other. To be a *complete basis*, the set has to have enough parameters to cover all the possible dimensionalities. So:

A *basis* is a set of parameters which are mutually independent.
A complete basis is a basis of $N$ parameters, where $N$ is the order of the class. (For example, the $LTP$ class is of order 3, so a complete basis would be a basis of 3 parameters.)

### 4.3 Testing A Possible Basis

A complete basis must have exactly $N$ parameters, where $N$ is the order of the class. (For example, the $LPT$ class is of order 3.)

To test whether the set \( \{ q_1, q_2, q_3 \} \) forms a complete basis of order 3, form the matrix of exponents. That is, in the $LTP$ class:

\[
\begin{align*}
[q_1] &= L^{a_1} T^{b_1} P^{c_1} \\
[q_2] &= L^{a_2} T^{b_2} P^{c_2} \\
[q_3] &= L^{a_3} T^{b_3} P^{c_3}
\end{align*}
\]

and the array of exponents would be

\[
\begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{pmatrix}
\]

If \( \|\mathbf{M}\| \neq 0 \), then the basis is complete (that is, mutually independent).

If, on the other hand, \( \|\mathbf{M}\| = 0 \), then at least two of the parameters in the basis are mutually dependent.

So let’s work some examples. Consider the set of three parameters $m$ (a mass), $g$ (gravitational acceleration), and $l$ a length. In the $MLT$ class, the array of exponents is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -2 \\
0 & 1 & 0
\end{pmatrix}
\]

whose determinant is not zero, so the set is dimensionally independent. By contrast, this set is not independent: $\omega$ (a frequency), $g$ and $l$, which can be seen from the vanishing determinant of its matrix:

\[
\begin{pmatrix}
0 & 1 & -1 \\
0 & 1 & -2 \\
0 & 1 & 0
\end{pmatrix}
\]
4.4 Analogy to Vectors and Vector Spaces

If you haven’t already noticed, this all looks very much like the idea of vectors and vector spaces. Let’s explore this analogy. Each basic type of dimensionality is like another direction in a space. Consider the \( LTP \) class, for example, where there are three independent types of dimensionality. This is analogous to a three-dimensional vector space. The dimensionality of a quantity \( q \) can be expressed as a power-law monomial

\[
[q] = L^a T^b P^c
\]

and the exponents \( \{a, b, c\} \) are analogous to an ordered triplet describing the components of a Cartesian vector. Thus, each parameter can mapped onto a vector in this abstract space, at least in terms of its dimensionality.

In this way, we can see that a single parameter (if not dimensionless) defines a “vector” in this space. (A dimensionless parameter would correspond to a vector of length 0). If two parameters are dimensionally dependent, then their “vectors” will be parallel.

Take the case where one has two dimensionally independent parameters. Their “vectors” form a “plane”. A third parameter would be dependent or independent on the other two if it lies in that plane or not.

Finally, there can be only as many independent parameters as their are directions in this vector space, so the largest set would be equal in number to the order of the class.

The test for independence given above, based as it is on the determinant of the vector components, is the same as computing the “volume” spanned by the vectors. For the vectors to be independent, the volume must not be zero; thus the determinant must not be zero.

4.5 Dimensionless Forms

If we have a complete basis, then an additional parameter must be dependent on that basis. This means, as we shall demonstrate now, that there is a dimensionless ratio formed from that additional parameter and powers of the basis parameters.

Let’s take a concrete example to show how this works. Suppose we have a complete basis, of \( N = 2 \) parameters (so the class is of order 2). Let’s
call the basis \( \{q_1, q_2\} \). Now those two parameters completely span the space (because the basis is complete). A third parameter must be dependent on those two, which means that there must exists numbers \( a \) and \( b \) such that

\[
[pq_1^a q_2^b] = 1
\]  

(41)

In other words, we can define a dimensionless form of \( p \), which we call \( \tilde{p} \), given by

\[
\tilde{p} = pq_1^a q_2^b
\]  

(42)

This can always be done. We can express all parameters outside of a complete basis in dimensionless form. The dimensionless form is just a number, and is independent of the system of units.

The importance of the dimensionless form will be made clear when we consider the Buckingham Π Theorem.

### 4.6 Examples

Consider this basis: \( m, g, v_0 \) (mass, acceleration, velocity) and the parameter \( t \) (time). We are looking for exponents \( a, b, \) and \( c \) to make the dimensionless parameter

\[
\tilde{t} = tm^a g^b (v_0)^c
\]  

(43)

dimensionless. On the RHS, in the \( MLT \) class

\[
[tm^a g^b (v_0)^c] = M^a L^{b+c}T^{1+2b-c}
\]  

(44)

so that we have three equations: \( a = 0, b + c = 0, \) and \( 1 - 2b - c = 0, \) from which we get \( a = 0, b = 1, \) and \( c = -1. \) That means that \( \tilde{t} = gt/v_0 \) is the only dimensionless combination of those parameters which is linear in \( t. \)
5 Buckingham Π Theorem

5.1 An example

Suppose we consider the case as we did before of a class of order 2, where we have designated two dimensionally independent parameters \(\{q_1, q_2\}\) as the basis. A third parameter \(p\) must be dimensionally dependent on the basis, and so we can define the dimensionless form of \(p\), call it \(\tilde{p}\), by:

\[
\tilde{p} = p q_1^a q_2^b \tag{45}
\]

where \(a\) and \(b\) have been determined.

Note here a remarkable thing. The dimensionless form \(\tilde{p}\) is just a number, and is independent of any change of units. Therefore we have the remarkable assertion:

\[
\tilde{p} = \Phi \tag{46}
\]

where \(\Phi\) is a constant.

This has remarkable consequences. Take a simple pendulum of mass \(m\) and length \(l\) swinging in a gravitational field of acceleration \(g\). We want to relate the frequency \(\omega\) of motion to those basic parameters. In the \(MLT\) class, we can pick three parameters to be the basis, and we pick those to be \(m\), \(g\), and \(l\). The dimensionless form of the remaining parameter \(\omega\) is then \(\tilde{\omega} = \omega \sqrt{l/g}\). We notice now that this must be a constant because it is independent of units, from which we conclude that

\[
\omega = \text{constant} \times \sqrt{g/l} \tag{47}
\]

The constant cannot be determined from dimensional analysis alone. Note that we have not made a small angle approximation!

5.2 Statement of the Theorem

**Buckingham Π Theorem:** If a system has \(n\) physical quantities of relevance that depend on \(k\) independent dimensions, then there are a total of \(n - k\) remaining dimensionless products \(\pi_1, \pi_2, \ldots, \pi_{n-k}\). The behavior of the system is describable by a dimensionless equation

\[
F(\pi_1, \pi_2, \ldots, \pi_{n-k}) = 0
\]
5.3 A Nice, Simple Recipe for Applying the Theorem

1. List all relevant quantities \( \{a_1, a_2, \ldots, a_n\} \) involved in the problem. \( n \) = number of them. (BE CAREFUL! Beginners often err by omitting a relevant parameter!)

2. Pick an appropriate class of units. \( k \) = size of the class (for example, LMT has \( k = 3 \)).

3. Express dimensionality of each quantity in the class.

4. Divide the set of quantities into two subsets. The first subset is the list of \( k \) quantities \( \{b_1, b_2, \ldots, b_k\} \), called the “basis”. The basis set must be dimensionally independent! The second subset is the remaining \( n - k \) quantities \( \{c_1, c_2, \ldots, c_{n-k}\} \), which are called “residuals”.

5. For each residual quantity \( c_m \), form a corresponding dimensionless quantity \( \tilde{c}_m \) using the basis quantities in the denominator:

\[
\tilde{c}_m = \frac{c_m}{b_1^{\alpha_1} b_2^{\alpha_2} \ldots b_k^{\alpha_k}}
\]

where the exponents \( \{\alpha, \beta, \ldots, \mu\} \) are chosen for each residual quantity to make the ratio dimensionless.

6. The result is expressed a relationship among the dimensionless residual quantities:

\[
F(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{n-k}) = 0
\]

7. If the governing equation is known, then one can express the governing equation in terms of the dimensionless residual quantities. To do this, simply rename each of the residual quantities to its dimensionless name. The quantities in the basis are each set to one.

In this way, the problem has been reduced from \( n \) quantities to \( n - k \).
6 Examples of the Buckingham $\Pi$ Theorem

6.1 Relating Basic Quantities

A star undergoes a mode of oscillation where it expands and contracts while maintaining spherical symmetry. How does the frequency $\omega$ of oscillation depend upon the properties of the star? The first step is the identification of the physically relevant variables. Certainly the density $\rho$ and the radius $R$ are important; well also need the gravitational constant $G$ which appears in Newton's law of universal gravitation. We could add the mass $m$ to the list, but if we assume that the density is constant as a first approximation, then $m = \rho(4\pi R^3/3)$, and the mass is redundant.

We want to find $\omega$, so let's leave that out of the basis. We have then three parameters ($\rho$, $R$, and $G$) and a class (let's use $MLT$) of order 3. The basis parameters have these dimension: $[\rho] = ML^{-3}$, $[R] = L$, $[G] = M^{-1}L^3T^{-2}$.

(How do we get that last one? Remember that the gravitational energy between two masses is given by $-GM^2/r$, and then $G$ has to have dimensions to make this an energy.)

Is this a proper basis? The array of exponents has the form

$$\mathbf{M} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ -1 & 3 & -2 \end{pmatrix}$$

which has non-zero determinant, so we're OK.

So we want to determine $a$, $b$, and $c$ such that

$$[\omega] = T^{-1}$$

$$= [\rho^a R^b G^c]$$

$$= M^{a-c} L^{-3a+b+3c} T^{-2c}$$

from which we get

$$a - c = 0$$

$$-3a + b + 3c = 0$$

$$-2c = -1$$

and, solving gives $a = c = 1/2$, $b = 0$, so that

$$\omega = C \sqrt{G\rho}$$
Notice that the frequency of the oscillation is proportional to the square root of the density and independent of the radius. This conclusion is based on dimensional analysis and the assumption that the only relevant parameters are $\rho$, $R$, and $G$.

Here’s another example. Consider the governing equation of a damped, simple harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0$$

starting from rest at $x(0) = x_0$. The motion expressed in $x(t)$ is characterized by $x_0$ and the three physical parameters $m$ (mass), $b$ (linear drag coefficient), and $k$ (spring constant), so we would say, in general, that

$$x = \Psi(t, x_0, m, b, k)$$

The BP theorem allows us to use the information that we have about the dimensionality of the parameters. We have six parameters involved, and we can pick three as the basis. The art (that comes with experience) is in picking which three to be the basis. Another way, is to think of what we don’t want in the basis. We want to solve for $x(t)$, so neither $x$ nor $t$ should be in the basis. Now, let’s think about the motion. There will be some sort of oscillatory behavior, so we could pick $m$ and $k$ as part of the basis, and also there is the original displacement, which sets the scale. So let’s try $\{x_0, m, k\}$ as the basis. We have residual parameters $x$, $t$, and $b$, which are in dimensionless form

$$\tilde{x} = \frac{x}{x_0} \quad \tilde{t} = \frac{t}{\sqrt{k/m}} \quad \tilde{b} = \frac{b}{\sqrt{km}}$$

And the BP theorem gives

$$\Phi(\tilde{x}, \tilde{t}, \tilde{b}) = 0$$

or solving for $\tilde{x}$ we get

$$\tilde{x} = \zeta(\tilde{t}, \tilde{b})$$

where $\zeta$ is an unknown function. Substituting the original parameters in, we have

$$x = x_0 \zeta( t\sqrt{k/m}, b/\sqrt{km})$$

You might recognize $\sqrt{k/m} = \omega_0$ is the frequency of the undamped oscillator and $b/\sqrt{km} = Q^{-1}$ where $Q$ is the (dimensionless) quality factor for the
system. We don’t know $\zeta$ yet, and will need to solve the governing equation to get that, but what we have is useful. We know from the study of damped, harmonic oscillators that the $Q$ is a convenient description of the type of motion displayed by the system.

### 6.2 Simplifying Governing Equations

We can also apply this to the simplification of governing equations. Consider this differential equation which governs the velocity $v$ of a particle of mass $m$ in a viscous medium of linear drag coefficient $b$:

$$ m\dot{v} + bv = 0 \quad (58) $$

which governs the function $v(t)$ subject to the initial condition

$$ v(0) = v_0 \quad (59) $$

Let’s list the parameters: $m$, $v$, $t$, $b$, and $v_0$.

First we need the dimensionality of each parameters. We can obtain the dimensionality of $b$ from the governing equation (in the $MLT$ class):

$$ [b] = \left[ m\dot{v}/v \right] = MT^{-1} \quad (60) $$

Before we apply the Buckingham Π-Theorem, let us consider what we’re after. We are looking to relate $v$ to $t$, so let’s leave those two parameters out of the basis. That leaves $m$, $b$, and $v_0$. First, we should verify that this is a proper basis (it is). Then we look for the dimensionless forms of the residual parameters $v$ and $t$, which are

$$ \tilde{v} = v/v_0 \quad (61) $$

$$ \tilde{t} = bt/m \quad (62) $$

Applying now the BP Theorem, we have

$$ \Phi(\tilde{v}, \tilde{t}) = 0 \quad (63) $$
where the function $\Phi$ is not known. Written a different way, we could formally solve Eq. 63 for $\tilde{v}$ to get

$$\tilde{v} = f(\tilde{t})$$  \hspace{1cm} (64)

or adding back in the original parameters

$$v = v_0 f(bt/m)$$  \hspace{1cm} (65)

where $f$ is not known solely on the basis of DA.

We can go even further. We note that the basis parameters are independent. We could chose a system of units that make those parameters any value we like — wouldn’t setting them all to 1 be simplest?

Let’s see what this does to our governing equation. We can replace all of the basis parameters by 1, and all of the residual parameters by their dimensionless form. Thus,

$$\tilde{v}\'(\tilde{t}) + \tilde{v}(\tilde{t}) = 0$$  \hspace{1cm} (66)

which governs the function $v(t)$ subject to the initial condition

$$\tilde{v}(0) = 1$$  \hspace{1cm} (67)

In this dimensionless form, the governing equation is somewhat simpler, because there are now three fewer parameters to deal with. The solution is

$$\tilde{v}(\tilde{t}) = e^{-\tilde{t}}$$  \hspace{1cm} (68)

from which we can see that the equation $f$ from before is simply $f(z) = e^{-z}$.

So there is the lesson: we can set to unity as many parameters as there are different dimensional types in the class, and they must be independent parameters. Once we do that, we can re-write the governing equations in a simpler form.

Note, however, when we do this, we have lost the ability to check our solutions using DA.

### 6.3 Data Collapse

We can also use DA to reduce the amount of work required in doing experiments. (See lecture. I need an example here.)
7 This is great stuff! What could possibly go wrong?

We see there are two common uses for dimensional analysis.

First, the case where we have a list of parameters which can be related. If the list of parameters is not too long, one can apply the Buckingham Pi Theorem to demonstrate a useful relationship between the parameters. The danger comes in forming the list: do we have all of the important parameters which should be considered? If not, leaving off an important parameter can be lead to dopey results. This is the most common cause of error in using dimensional analysis. The difficulty comes in how we reduce the phenomenon or effect to a set of physical parameters. When doing this, there is no substitute for care, and for checking that the results are sensible. As I said earlier, this is probably the main reason why dimensional analysis is not taught more widely in physics courses.

A second use, is when we have the governing equation(s). To the extent that the governing equation is correct, it is safe to apply dimensional analysis, because we have a well-defined set of parameters involved.

There is one subtlety, which we don’t have time to consider here. Suppose that we have a set of parameters, what happens if one of the parameters were deliberately neglected? We usually can arrange this so that neglecting the parameter is equivalent to setting its value to zero. So, what happens when that parameter goes to zero? Two things can happen. One possibility is that nothing catastrophic happens; the functional dependence on that parameter is smooth, so that there is no qualitative or singular behavior as it goes to zero. The second possibility is that the functional behavior is not smooth, that there is a singular behavior in this limit. This latter case is a characteristic of some non-linear systems, and often leads to self-similar solutions which have a very interesting role in physics.

We don’t have time to consider this very interesting topic, but fortunately there is an excellent book on the subject. See “Scaling, self-similarity, and intermediate asymptotics” by Grigory Barenblatt (published in 1996 by Cambridge Texts in Applied Mathematics).