Homework 1 — Math 493

In order to obtain maximum marks you should show all your working clearly and concisely.

Question 1 Prove that the solution to the scalar initial value problem

$$u' - a(t)u = f(t), \quad u(0) = u_0$$

is

$$u(t) = \int_0^t e^{h(t)-h(s)} f(s)\, ds + e^{h(t)} u_0$$

where $h(t) = \int_0^t a(s)\, ds$. (Hint: start by multiplying both sides of the equation by the integrating factor $e^{-h(t)}$).

Question 2
a) Verify that $\{h_n\}_{n=0}^{n=\infty} = \{\cos(nx)\}_{n=0}^{n=\infty}$ is an orthogonal set on $[0, \pi]$.
b) For each $n = 0, 1, 2, \ldots$ calculate $\|h_n\|_{L^2}$. Note that your answers to a) and b) will need separate calculations for the case when $n = 0$.
c) Assuming that the function $f(x) = x - 2$ on the interval $[0, \pi]$ can be written as

$$x - 2 = \sum_{n=0}^{\infty} c_n \cos(nx),$$

find the coefficients c_n. Show how this sum converges to $x - 2$ on the interval $[0, \pi]$ by writing a computer program to plot $\sum_{n=0}^{M} c_n \cos(nx)$ for $M = 3, 10, 20$.

Question 3 Carefully stating which results from the classnotes you are using, find the solution $u(x,t)$ on the interval $x \in [0, 1], t > 0$ to the problem $u_t = u_{xx}$ with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition $u_0(x) = 1$.

Question 4 Prove that, under appropriate boundary conditions, any two eigenfunctions u_1 and u_2 of a Sturm-Liouville problem $(pu')' + gu + \lambda ru = 0$ that correspond to different eigenvalues λ_1 and λ_2 are orthogonal with respect to the weight function $r(x)$. (Hint: start by multiplying the S-L equation for u_1 by u_2 and vice versa. Then subtract one of these equations from the other and integrate over the interval upon which the problem is defined).

Question 5 The *Trapezoidal Rule* for solving a scalar IVP is given by

$$y_{n+1} = y_n + \frac{1}{2}\Delta t[f(y_n) + f(y_{n+1}].$$

Redo the stability analysis from class for the behaviour of this method on the linear scalar problem $y' = ay$, $Re(a) < 0$. What is the stability region, S. in the complex plane?