Math 493 **Final Exam** June 21, 2007

This final exam is closed book and closed notes, but you may use a calculator. Do all of the following six problems. (Do not forget the problems on the back!!)

(1) Use separation of variables to solve the equation $u_t(x,t) = 3u_{xx}(x,t)$ on the domain $0 < x < 5$, $t > 0$, with boundary and initial conditions

$$u_x(0,t) = 0\ , \quad u_x(5,t) = 0\ , \quad \text{and} \quad u(x,0) = 27\ .$$

(2) Let $u(x,t)$ and $v(x,t)$ both be solutions to the hyperbolic partial differential equation $\phi_{tt} = c^2\phi_{xx}$, on the domain $0 < x < 1$, and $t > 0$, with boundary conditions

$$\phi(0,t) = 9\sinh(t) \quad \text{and} \quad \phi(1,t) = t^3 + \sin(t)\ ,$$

and initial conditions

$$\phi(x,0) = x^2 - 3x + 5 \quad \text{and} \quad \phi_t(x,0) = x^5 - 4\ .$$

Show that $v(x,t) = u(x,t)$ for all $0 < x < 1$, and $t > 0$.

Hint: Find a PDE problem with boundary and initial conditions solved by the function $w(x,t) = u(x,t) - v(x,t)$. Consider the integral

$$F(t) = \int_0^1 \left((w_t(x,t))^2 + c^2\,(w_x(x,t))^2 \right)\, dx\ .$$

(3) Consider the finite difference formula for the third derivative of a function $u(x)$ with respect to x at a point x_j

$$u_j''' = \frac{1}{2h^3}\left[u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}\right]$$

where h is the spacing between points (e.g. $x_j = (j-1)h$) and u_{j+2} approximates the function value $u(x_j + 2h)$, etc.

(a) The Taylor expansion for u_{j+1} can be expressed as

$$u_{j+1} = u_j + hu_j' + \frac{h^2}{2!}u_j'' + \frac{h^3}{3!}u_j''' + \frac{h^4}{4!}u_j^{(iv)} + \frac{h^5}{5!}u_j^{(v)} + O(h^6)$$

where $u_j^{(iv)}$ represents the fourth derivative, for example. Write out similar expansions for u_{j-1}, u_{j-2} and u_{j+2}.

(b) Use the results of part (a) to show that the above finite difference formula for u_j''' is second order accurate (i.e. the associated error is $O(h^2)$).

(4) Consider the partial differential equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

subject to the condition that $u(x, t = 0) = x$ where $a > 0$ is a constant.

A numerical approach to solving this problem is the Lax-Wendroff scheme

$$u_j^{i+1} = \frac{1}{2}ap(1 + ap)u_{j-1}^i + (1 - a^2p^2)u_j^i + \frac{1}{2}ap(-1 + ap)u_{j+1}^i \,,$$

where the superscript is the index in time and the subscript is the index in space. That is, $x_j = (j-1)h$ where h is the space step and $t_i = (i-1)k$ where k is the time step. Here $p = k/h$.

(a) Which values u_j^1 (i.e. which values of j) eventually influence the numerically computed value u_4^3?

Hint: Drawing a grid in x and t may be helpful. You are not required to actually compute u_4^3.

(b) Explain what the CFL condition implies in this context.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ denote a 1-periodic function, and consider the discretization points $x_k = k/N$, where $k = 0, \dots, N-1$, for some positive integer N.

(a) Describe the method of spectral differentiation for computing approximations to the values $f'(x_k)$ from the values $f(x_k)$, where $k = 0, \dots, N-1$. Describe the general philosophy behind the algorithm, as well as the algorithmic implementation.

(b) For positive integers ℓ, consider the functions $f(x) = |\sin(2\pi x)|^\ell$. Which accuracy of the approximation in (a) do you expect? Your answer should depend on the exponent ℓ. Cite any theorems from class that you use.

(6) Consider the elliptic problem $y'' + y = t$ with initial conditions $y(0) = 0$ and $y'(0) = 2$. Consider the basis functions $\phi_k(t) = t^k$, and let $\hat{y}(t) = \sum_{k=0}^{N} c_k \varphi_k(t)$ for coefficients $c_k \in \mathbb{R}$.

Compute an approximation $\hat{y}$ to the solution y of the above elliptic problem using the collocation method with $N = 3$ and the collocation points $t_1 = 1/2$ and $t_2 = 1$.