Differential Geometry Seminar Notes

Book: *Elementary Differential Geometry* by O'Neill

George Mason University

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Contents

1 Lecture 1 - Ryan Vaughn - 02/01/2018

Definitions and Notations:

1. $\mathbb{R}^n$ - n-tuples of points $p = (p_1, ...p_n)$ such that $p_i \in \mathbb{R}$. $\mathbb{R}^3$ we distinguish $p = (p_1, p_2, p_3) \in \mathbb{R}^3$,
2. $x : \mathbb{R}^3 :\to \mathbb{R}$ where $x(p) = p_1$ are the coordinate functions.
3. $C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}$: all partial derivatives of f exist of all orders and are continuous for all $p \in \mathbb{R}^n\}$.

Example 1.1. Consider the following function f.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{at } (x,y) = (0,0) \end{cases}$$

Then f is an example where all partial derivatives exist but the total derivative doesn't.

Recall:

$$\left.\frac{\partial f}{\partial x_i}\right|_{p \in \mathbb{R}^n} = \lim_{t \to 0} f \frac{(p + te_i) - f(p)}{t}$$

Definition 1.2. Let $p \in \mathbb{R}^3$, the tangent space at p, $T_p\mathbb{R}^3$ is the set

$$T_p(\mathbb{R}^3) = \{v_p = (v, p) : v \in \mathbb{R}^3\}$$

Definition 1.3. *A vector field V is a function which maps each point p to a tangent vector v_p.*

$$V : \mathbb{R}^3 \to T\mathbb{R}^3 = \bigcup_{p \in \mathbb{R}^3} T_p\mathbb{R}^3$$

Vector fields can be added and multiplied by $C^\infty(\mathbb{R}^n)$ functions.

$$v + w(p) = v(p) + w(p) \;\forall\; p \in \mathbb{R}^3.$$

$$fv(p) = f(p)v(p)$$

Example 1.4. $u_1(p) = (1,0,0)|_p$ $u_2(p) = (0,1,0)|_p$, $u_3(p) = (0,0,1)|_p$. At each point p, $\{u_1(p), u_2(p), u_3(p)\}$ form a basis for $T_p(\mathbb{R}^3)$. A *frame* is a collection of vector fields that form a basis at each point for the vector fields. This is called the *natural frame field.* It is a global frame).

Remark: Sometimes tangent vectors are defined as being linear derivations(Lee).

Lemma 1.5. *If V is a vector field in $\mathbb{R}^3$, then V can be written uniquely as*

$$V = v_1u_1 + v_2u_2 + v_3u_3 \textit{ where } v_1, v_2, v_3 : \mathbb{R}^3 \to \mathbb{R}.$$

Here v_1, v_2, v_3 are called Euclidean Coordinate Functions.

Smooth or continuous vector fields:

A vector field is *smooth* iff $v_1, v_2, v_3 \in C^\infty(\mathbb{R}^3)$.
A vector field is *continuous* iff $v_1, v_2, v_3 \in C^0(\mathbb{R}^3)$.

1.1 Directional derivatives :

Definition 1.6. Directional derivatives take in tangent vectors and functions $C^\infty(\mathbb{R}^3)$ and output a real number. Directional derivatives of f in the direction v at point p is denoted

$$v_p(f) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t} = \frac{d}{dt} f(p+tv)\bigg|_{t=0}.$$

Lemma 1.7 (pg 12). *Let $v_p = (v_1, v_2, v_3)_p$, then*

$$v_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p) = v_p \cdot \bigtriangledown f|_p = \left(\frac{\partial f}{\partial x_i}\bigg|_p \right).$$

Remark: Directional derivative is a linear derivation on $C^\infty(\mathbb{R}^3)$.

Theorem 1.8. *Let $f, g \in C^\infty(\mathbb{R}^3)$ and $v_p, w_p \in T_p(\mathbb{R}^3)$ and $a, b \in \mathbb{R}$*

$$1) v_p[af + bg] = av_p[f] + bv_p[g] (\text{ linear })$$

$$2) v_p[fg] = v_p[f]g(p) + f(p)v_p[g] (\text{ derivation })$$

$$3) av_p + bv_p[f] = av_p[f] + bw_p[f]$$

2 Lecture 2 - Ryan Vaughn - 01/07/2018

The talk started with highlights from last lecture.

2.1 Curves in $\mathbb{R}^3$

Section 1.4 from textbook.

Definition 2.1. A curve $\alpha : I \to \mathbb{R}^3$ is a *smooth function* where I is an open interval .

Example 2.2 (Straight line.)**.** Let $p, q \in \mathbb{R}^3$.

$$\alpha(t) = p + tq.$$

The straightline is based at p in the direction q.

Big point: given a tangent vector v_p there exists a straight line

$$t \mapsto p + tv$$

such that $\alpha(0) = p$.

Every curve has a natural tangent vector“at each point” along the curve.

Definition 2.3. Velocity of α at t_0

$$\alpha(t_0) = \left(\frac{d\alpha_1}{dt}, ..., \frac{d\alpha_n}{dt}\right)\Bigg|_{\alpha(t_0)}$$

The velocity vector corresponds to a tangent vector V_p at a base point p.

Definition 2.4. Let J be an open interval $h : J \to I$ be smooth. We call the curve $\beta = \alpha \circ h$ a *reparametrization* of α. By the chain rule

$$\beta'(t) = \frac{dh}{dt}\bigg|_t \cdot \alpha'(t)$$

Note: Another way of defining tangent vectors is as an equivalence class of curves

$$V_p = [\alpha].$$

where

1. $\alpha(0) = p$.
2. $\alpha'(0) = v$
3. $\alpha \sim \beta$ if $\alpha'(0) = \beta'(0)$ i.e., $[\alpha] = [\beta]$.

2.2 Mappings

Section 1.7 from textbook.

Overview: Every *smooth function* (mappings) $F : \mathbb{R}^n \to \mathbb{R}^m$ induces linear maps

$$F_{*_p} : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m.$$

Definition 2.5. Let F be as above. If $v_p \in T_p(\mathbb{R}^n)$, then let $F_{*_p}(v_p)$ be the initial velocity of the curve $t \mapsto F(p + tv)$.

Proposition 2.6. *Let $F = (f_1, .., f_m), F : \mathbb{R}^n \to \mathbb{R}^m$. Then*

$$F_*(v_p) = (v_p[f_1], .., v_p[f_m]) = (v[f_1], .., v[f_m])\Big|_p \text{(the directional derivative at } p\text{)} .$$

Proof. Let

$\beta = F(p + tv)$.

$$\begin{aligned} F_{*_p}(v_p) = \beta'(0) &= \frac{d}{dt}\Big(f_1(p+tv), ..., f_m(p+tv)\Big) \\ &= \left(\frac{d}{dt}f_1(p+tv), ..., \frac{d}{dt}f_m(p+tv)\right) \\ &= \Big(v_p[f_1], .., v_p[f_m]\Big) \end{aligned}$$

□

Exercise 1. *What is $F_*(e_1|_p)$ where $e_1 = (1, 0, 0, ..., 0)$?*

$$F_*(e_{1_p}) = \begin{pmatrix} e_1[f_1] \\ \vdots \\ e_1[f_n] \end{pmatrix}\Bigg|_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(p) \end{pmatrix}$$

This exercise clearly generalizes to the i-th standard basis vector e_i. Simply substitute i for 1 in the above.

Writing F_* as a matrix. Choose $(e_i)|_p$ for $i \in \{1, ..., n\}$ as a basis for $T_p(\mathbb{R}^n)$, and for $T_{F(p)}\mathbb{R}^m(e_j)|_{F(p)}$ for $j \in \{1, ..., m\}$. We know

$$e_i \mapsto \begin{pmatrix} \vdots \\ \frac{\partial f_j}{\partial x_i} \\ \vdots \end{pmatrix}$$

The ith column of our matrix should be $\begin{pmatrix} \vdots \\ \frac{\partial f_j}{\partial x_i} \\ \vdots \end{pmatrix}$.

Matrix representation of F_* at p is the Jacobian J at p.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \dots & \frac{\partial f_m}{\partial x_n}(p) \end{pmatrix}$$

Definition 2.7. A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is regular if F_{*_p} is one to one for all points $p \in \mathbb{R}^n$.

From linear algebra, the following are equivalent.

1. F_{*_p} is one to one.
2. $F_{*_p}(v_p) = 0$ if and only if $v_p = 0$.
3. The Jacobian at p is rank n where n is the dimension of the domain in F.

Theorem 2.8 (Inverse Function Theorem). *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If F_{*_p} is one to one, there exists open set $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ such that $F|_u : U \to V$ is bijective, smooth, and $F^{-1}|_V V \to U$ is smooth. In other words, $F|_U$ is a diffeomorphism.*

Exercise 2.

1. Anything in 1.7. If you choose to do any of 1-4, try to do all of 1-4, since they are related.
2. Prove the Inverse Function Theorem for $\mathbb{R}^n \to \mathbb{R}^n$.
3. Prove that the following three definitions of $T_p(\mathbb{R}^n)$ are isomorphic as vector spaces over $\mathbb{R}$
   (a) Set of ordered pairs (V, p) in$\mathbb{R}^n \times \{p\}$.
   (b) Set of linear derivations over $\mathbb{R}$ in $C^\infty(\mathbb{R}^n)$.
   (c) Set of equivalence classes of curves with the same initial velocity.

3 Lecture 3 -Problem discussion - 02/14/2018

Exercise 3 (problem 7.6 from book - presented by Michael)**.** *a. Give an example to demonstrate that a one-to-one and onto mapping need not be a diffeomorphism.*

b. Prove that if a one-to-one and onto mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is regular, then it is a diffeomorphism.

Proof. a. $f : \mathbb{R} \to \mathbb{R}$. $f(x) = x^3$. Then $f^{-1}(x) = x^{1/3}$ and $(f^{-1})'(x) = -\frac{1}{3}x^{\frac{-2}{3}}$ is continuous but not.

b. $F : \mathbb{R}^n \to \mathbb{R}^n$ is regular and bijective. Then it has a set theoretic inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. F_{*_p} is injective for all $p \in \mathbb{R}^n$ by regularity. By inverse function theorem, for any $p \in \mathbb{R}^n$, there exists open $U \subset \mathbb{R}^n$ such that $F|_U$ is a diffeomorphism onto its image. $F^{-1}|_{F(U)}$ is differentiable, so F^{-1} is differentiable everywhere.

□

Exercise 4. *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, prove that $F_*(v_p) = F(v)_{F(p)}$.*

Proof. Let $F = (f_1, ..., f_m)$. Then $F(v) = Av$ where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

Then $f_i(v) = a_{i1}v_1 + \ldots + a_{in}v_n$. Therefore,

$$\frac{\partial f_i}{\partial x_1} = a_{i1}$$

. Therefore

$$F_*(v_p) = J_p(v_p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = A$$

□

4 Lecture 4 - Ryan Vaughn - 02/21/2018

4.1 The Covariant Derivative in $\mathbb{R}^3$

Note: This chapter heavily relies on properties of $\mathbb{R}^3$ as opposed to general manifold M of dimension n.

1. $\mathbb{R}^3$ has a global chart
2. $\mathbb{R}^3$ is parallelizable. It has a global orthonormal frame field. u_1, u_2, u_3 such that all vector fields can be written

$$V = f_1 u_1 + f_2 u_2 + f_3 u_3 \text{ and}$$

$$u_i|_p \cdot u_j|_p = \delta_i^j (\text{ orthonormal})$$

M is parallelizable means $TM \cong M \times \mathbb{R}^n$ and $1 \implies 2$.
Why is parallelizable good?
It allows us to express

$$V : M \to TM$$

$$V = v_1 u_1 + v_2 u_2 + v_3 u_3 \text{ for } v_1, v_2, v_3 \in C^\infty(M)$$

$$V \cong (v_1, v_2, v_3) \in C^\infty(M, \mathbb{R}^n)$$

If M is globally covered, $M \xrightarrow{\phi} \mathbb{R}^n$ which is diffeomorphism.

Thought Experiment: Suppose $M = S^2$. Let V be a vector field on S^2.

Theorem 4.1. *Every vector field on S^2 has a point $p \in S^2$ such that $W(p) = 0$.*

An immediate consequence of the theorem is that we cannot have a global orthonormal frame and hence S^2 is not parallelizable becaue there will always be atleast one point where W is zero and hence can't span the whole of S^2. You can always construct local frames:
Let (U, ϕ) be a chart of S^2 about p since

$$\phi : U \to \mathbb{R}^2$$

is a diffeomorphism

$$d\phi_p : T_p U \to T_{\phi(p)} \mathbb{R}^2$$

is a linear isomorphism for all $p \in U$. So we can define a pullback frame $\phi^*(U_i)$ by

$$\phi^*(U_i)_p = d\phi^{-1}_{\phi(p)} U_i(p)$$

where $U_i(p)$ is a vector in $\mathbb{R}^2$ and $d\phi^{-1}_{\phi(p)}$ is a linear isomorphism $\mathbb{R}^2 \cong T\mathbb{R}^2 \to T_p M$.
So we can't write W as a vector field on S^2 as $W = w_1 u_1 + w_2 u_2$ with u_1, u_2 vector fields on S^2.

Definition 4.2. Let $p \in \mathbb{R}^3$ and V, W be tangent vector fields. The covariant derivative of W with respect to V is the tangent vector

$$\triangledown W|_p = (\overline{W}(p+tv))'(0) = \overline{W}_*(v_p)$$

where $\overline{W} = (w_1(p), w_2(p), w_3(p))$ and $W = w_1u_1 + w_2u_2 + w_3u_3$.

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$\triangledown : \text{Vector Fields(input)} \ \times \ \text{Vector Fields(direction)} \ \to \ \text{Vector Fields}$$

Measures initial rate of change of W in the direction of V.

Example 4.3. Let

$$W = x^2u_1 + yzu_2.$$

$v = (-1, 0, 2)$ at $p = (2, 1, 0)$. Then

$$\triangledown_v W\Big|_p = (-4, 2, 0)\Big|_{(2,1,0)}$$

$$p + tv = (2-t, 1, 2t)$$

Then $w_1 = x^2, w_2 = yz, w_3 = 0$. Therefore,

$$\overline{W}(p+tv) = \Big((2-t)^2, 2t, 0\Big)$$

$$\begin{aligned}\Big(\overline{W}(p+tv)\Big)'(0) &= \Big(-2(2-t), 2, 0\Big) \\ &= (-4+2t, 2, 0)\Big|_{t=0} \\ &= (-4, 2, 0)\end{aligned}$$

Theorem 4.4. *Let W, V, Y, Z be vector fields to $\mathbb{R}^3$ and let $f, g \in C^\infty(\mathbb{R}^3)$*

1. $\triangledown_{fV+gW} Y = f \triangledown_V Y + g \triangledown_W Y \triangledown$
2. $\triangledown_V (fY + gZ) = f \triangledown_V Y + g \triangledown_V Z$
3. $\triangledown_r fY = V[f]Y(p) + f(p) \triangledown_V Y$
4. $v[Y \cdot Z] = \triangledown_V Y \cdot Z(p) + Y(p) \triangledown_V (Z)$

Lemma 4.5. *Let $W = \sum w_i u_i$ be a vector field on $\mathbb{R}^3$, and let $v_p \in T_p\mathbb{R}^3$, then*

$$\triangledown_V W = \sum_{i=1}^{3} v[w_i]u_i(p)$$

Proof. Recall $V[f] =$ "*grad* $f \cdot V$" $= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} V_j$. So

$$\sum_{i=1}^{3} v[w_i]u_i(p) = \sum_{i=1}^{3}\sum_{j=1}^{n} \frac{\partial w_i}{\partial x_j} v_j u_i(p) = D\overline{W}\Big|_p \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \overline{W}_*\Big|_p (v_p) = \triangledown_V W\Big|_p$$

$\square$