Differential Geometry Reading Group Exercises

February 21, 2018

Chapter 1

Book Exercises

Extra Problems

Note that in these exercises, we denote the i-th projection function by using a superscript x^i instead of x_i. This is a convention in some differential geometry textbooks, because it allows us to use Einstein notation, which is useful when you deal with lots of sums.

1. Let $p \in \mathbb{R}^n$ be fixed and define $T_p\mathbb{R}^n = \{v_p = (v, p) : v \in \mathbb{R}^n\}$. We see that $T_p\mathbb{R}^n$ is a vector space of dimension n by addition and scalar multiplication of vectors.

   Let $\mathcal{V}_p$ be the set of functions $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that for any $c \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned} v(cf + g) &= cv(f) + v(g) \\ v(fg) &= f(p)v(g) + v(f)g(p). \end{aligned}$$

   The set $\mathcal{V}_p$ is also a vector space, since real-valued functions can be added and multiplied by a scalar.

   Show that $T_p\mathbb{R}^n$ and $\mathcal{V}_p$ are isomorphic as vector spaces.

   Hint: Note that $\frac{\partial}{\partial x^i} \in \mathcal{V}_p$. Show that the set $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is a basis for $\mathcal{V}_p$. Then construct the linear isomorphism by mapping $e_i \mapsto \frac{\partial}{\partial x^i}$ and extending linearly.

Proof. We first show that the set $X = \left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ is linearly independent in $\mathcal{V}_p$. Let $c_1, ..., c_n \in \mathbb{R}$ such that

$$c_1 \frac{\partial}{\partial x^1} + ... + c_n \frac{\partial}{\partial x^n} = \mathbf{0}.$$

This means that for all $f \in C^\infty(\mathbb{R}^n)$:

$$c_1 \frac{\partial f}{\partial x^1} + ... + c_n \frac{\partial f}{\partial x^n} = 0.$$

Recall that for each $i \in \{1, ..., n\}$ we have the i-th projection function $x^i : \mathbb{R}^n \to \mathbb{R}$ mapping a vector $p \in \mathbb{R}^n$ to it's i-th component value. If we plug in $x^1 \in C^\infty(\mathbb{R}^n)$ to the above, we obtain:

$$\begin{aligned} c_1 \frac{\partial x^1}{\partial x^1} + ... + c_n \frac{\partial x^1}{\partial x^n} &= 0 \\ c_1 \frac{\partial x^1}{\partial x^1} + 0 + ... + 0 &= 0 \\ c_1 + 0 + ... + 0 &= 0. \end{aligned}$$

Noticing that this argument holds for any x^i, we can repeat the procedure for all $i \in \{1, ..., n\}$ to show that $c_i = 0$ for all $i \in \{1, ..., n\}$. Hence, if there exists a collection of c_i's for which

$$c_1 \frac{\partial}{\partial x^1} + ... + c_n \frac{\partial}{\partial x^n} = \mathbf{0},$$

then they all must be zero and so the set X is linearly independent.

Now, we must show that X spans $\mathcal{V}_p$. To do so, we use Taylor's theorem in $\mathbb{R}^n$, as well as an additional lemma (both of which are proven in Lee's Smooth Manifolds.)

First, we set some notation. Let $I = (i_1, ..., i_m)$ be a m-tuple of indices with $1 \leq m \leq n$ and we say $|I| = m$ to denote the number of indices in I. Define

$$\partial_I = \frac{\partial^m}{\partial x^{i_1}, ..., \partial^{x^i_m}}$$

$$(x-a)^I = (x^{i_1} - a^{i_1})...(x^{i_m} - a^{i_m})$$

Theorem 1. *Let $U \subseteq \mathbb{R}^n$ be an open subset and let $a \in U$ be fixed. suppose that $f \in C^{k+1}(U)$ for some $k \geq 0$. If W is any convex subset of U containing a, then for all $x \in W$,*

$$f(x) = P_k(x) + R_k(x)$$

where P_k is the ***k-th order Taylor polynomial of f at a****, defined by:*

$$P_k(x) = f(a) + \sum_{m=1}^{k} \frac{1}{m!} \sum_{I:|I|=m} \partial_I f(a)(x-a)^I,$$

*and R_k is the k****-th remainder term,*** *given by*

$$R_k(x) = \frac{1}{k!} \sum_{I:|I|=k+1} (x-a)^I \int_0^1 (1-t)^k \partial_I f\big(a + t(x-a)\big)dt.$$

Lemma 1. *Let $p \in \mathbb{R}^n$, $v \in \mathcal{V}_p$, and $f, g \in C^\infty(\mathbb{R}^n)$.*

1. *If f is a constant function, then $v(f) = 0$.*
2. *If $f(p) = g(p) = 0$, then $v(fg) = 0$*

Proof. It suffices to prove i for the constant function $f_1(x) \equiv 1$, for then $f(x) \equiv c$ implies $v(f) = v(cf_1) = cv(f_1) = 0$. For f_1, we have:

$$v(f_1) = v(f_1 f_1) = f_1(p)v(f_1) + v(f_1)f_1(p) = 2v(f_1)$$

and so $v(f_1) = 0$. Similarly, for ii, we use the product rule:

$$v(fg) = f(p)v(g) + v(f)g(p) = 0 + 0 = 0.$$

□

Now, to show that X spans $\mathcal{V}_p$, we let $v \in \mathcal{V}_p$, and let $f \in c^\infty(R^n)$. Then by Taylor's theorem for $k = 1$ and $a = p$, we have that

$$v(f) = v(P_1 + R_1) = v(P_1) + v(R_1).$$

Notice that

$$R_1 = \sum_{i,j=1}^{n} (x^i - p^i)(x^j - p^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(p + t(x-p))dt$$

is the sum of products of two functions which are equal to 0 when we plug in $x = p$. The first function is

$$g_i(x) = x^i - p^i,$$

the second is

$$h_j(x) = (x^j - p^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(p + t(x-p))dt$$

times the integral. Note that for any i or j, if we plug in $x = p$ to g_i or h_j we obtain 0. Hence, using the second part of the lemma, we see that $R_1 = \sum_{i,j=1}^{n} g_i h_j$ for which $g_i(p) = h_j(p) = 0$ so $v(R_1) = 0$.

Now, looking at $v(P_1)$, we see that

$$\begin{aligned} v(P_1) &= v\left(f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) \right) \\ &= v\left(f(p)\right) + \sum_{i=1}^{n} v\left(\frac{\partial f}{\partial x^i}(p)(x^i - p^i) \right) \\ &= v\left(f(p)\right) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)v(x^i) - \frac{\partial f}{\partial x^i}(p)v(p^i) \\ &= 0 + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)v(x^i) - 0 \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)v(x^i) \end{aligned}$$

We have finally shown that $v(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)v(x^i)$. We simply re-interpret the sum, since $\frac{\partial}{\partial x^i}|_p$ is an element of $\mathcal{V}_p$, so we let $v(x^i)$ be the coefficients and $\frac{\partial}{\partial x^i}|_p$ be the elements of $\mathcal{V}_p$. Hence, we have:

$$v(f) = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i}\bigg|_p (f).$$

Since v is arbitrary, we see that the set $X = \left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^{n}$ spans $\mathcal{V}_p$ and thus is a basis. Hence, $T_p(\mathbb{R}^n) \cong \mathcal{V}_p$ by mapping

$$(e_i)_p \mapsto \frac{\partial}{\partial x^i}$$

and extending the map linearly. □