The Hodge Decomposition Theorem

Ryan Vaughn

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The Hodge Decomposition Theorem

**Theorem**
Let \((M, g)\) be a compact, Riemannian manifold. Then for each \(k = 1, \ldots, n\), the Hilbert space \(\Omega^k(M)\) of differential \(k\)-forms on \(M\) admits an orthonormal decomposition:

\[
\Omega^k(M) = \text{im} \, d \oplus \text{im} \, \delta \oplus \ker \Delta_k.
\]

Where \(d\) denotes the exterior derivative, \(\delta\) denotes the codifferential, and \(\Delta_k = d\delta + \delta d\) is the Hodge Laplacian on \(M\).

**Corollary**
\(H^k_{dR}(M) \cong \ker(\Delta_k)\)
Motivation

- Topological information (De Rham Cohomology Groups) of $M$ can be inferred by knowing the kernel of the $k$-Laplacian $\Delta_k$.
- In the case $k = 0$ (and sometimes $k = 1$), the spectrum of $\Delta_k$ can be inferred from finite data sampled from $M$.
- One of the goals for my dissertation is to find a way to infer for $k > 0$. 
Overview

- Introduction to differential forms.
- Define common operations $\wedge$, $\ast$, on differential forms.
- Define the exterior derivative $d$ as well as the codifferential $\delta$.
- Construct the Hodge Laplacian $\Delta = d\delta + \delta d$.
- Define the De Rham Cohomology groups $H^k(M)$.

We will see how the Hodge Decomposition Theorem tells us that

$$H^k_{dR}(M) \cong \ker(\Delta_k).$$
Let $M$ be a smooth manifold of dimension $n$.

- Hausdorff, second countable topological space that is locally homeomorphic to $\mathbb{R}^n$.
- Equipped with a *smooth structure*, so one can define smooth functions on $M$.

Examples: $\mathbb{R}^n$, spheres, torus
For each point $p \in M$, one can define the *tangent space at p* denoted $T_pM$, which is a dimension $n$ vector space.

Given a smooth map $f : M \to N$, we obtain linear maps $df_p : T_pM \to T_{f(p)}N$ on each of the tangent spaces.
A Riemannian manifold \((M, g)\) is a smooth manifold together with a choice of inner product \(\langle \cdot, \cdot \rangle_g\) on each tangent space \(T_pM\).

This allows us to measure lengths and angles of vectors in each tangent space, as well as the lengths of curves \(\gamma : [a, b] \rightarrow M\).
Let $M$ be a smooth manifold of dimension $n$:

- A differential $n$-form on $M$ is like choosing a determinant on each tangent space $T_pM$.

\[
\det\left(\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\right) = \text{Volume}
\]

- The determinant is \textit{multilinear}.
- The determinant is \textit{alternating}: If I plug in a set of linearly dependent vectors $v_1, \ldots, v_n$, then the determinant is zero.
Let $M$ be a smooth manifold of dimension $n$:

- Formally, a differential $n$-form $\omega$ is a choice of alternating, multilinear map

$$\omega_p : T_p M \times \ldots \times T_p M \to \mathbb{R}$$

for each $p \in M$. Meaning that $\omega_p$ is zero whenever a linearly dependent set of vectors is inputted. We also require $\omega_p$ to vary smoothly across $M$ a function of $p$.

- Intuitively, this is a way to measure “volume” in each tangent space.
Now let \((M, g)\) be an (orientable) Riemannian manifold of dimension \(n\).

- There is an obvious choice for an \(n\)-form.
- If we take an orthonormal set \(\{v_1, \ldots, v_n\}\) of vectors in \(T_pM\), the hypercube spanned by the vectors “should” have volume 1.
- There exists a unique \(n\)-form on \(M\) with this property, called the \textit{Riemannian volume form}, denoted \(V_g\).
- \(V_g\) is the differential form that assigns unit volume to unit hypercubes in each \(T_pM\).
Now \( M \) be a manifold of dimension \( n \) and let \( k < n \).

- Let \( \{e_1, ..., e_n\} \) be a basis for \( T_pM \).

- **Example:** Let \( v_1, v_2 \in T_pM \) and Denote \( e^1 \wedge e^2(v_1, v_2) \) as the 2-d volume of the square formed by projecting \( v_1 \) and \( v_2 \) onto the subspace formed by \( e_1 \) and \( e_2 \).

- \( e^1 \wedge e^2 \) is a 2-form. (Assuming we construct it across all \( T_pM \).)
For any choice of \( k \) unique basis vectors \( \{e_{i_1}, ..., e_{i_k}\} \), I can define \( e^{i_1} \wedge ... \wedge e^{i_k} \) in the same manner to measure the \( k \) dimensional area of vectors in the subspace spanned by \( e_{i_1}, ..., e_{i_k} \).

- For technical reasons, we always choose \( i_k \) to be strictly increasing.

- Each such \( e^{i_1} \wedge ... \wedge e^{i_k} \) can be added and scalar multiplied.
A differential $k$-form $\omega$ can be written as:

$$\omega = \sum_{I=(i_1,...,i_k)} f_I e^I$$

where $e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$ and $f_I : M \to \mathbb{R}$ is smooth. (The $f_I$ represent a choice of linear constants for each fixed $T_p M$ that varies smoothly across $M$.)

The set of all differential $k$-forms on a manifold $M$ is denoted $\Omega^k(M)$. 

$k$-forms

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The Hodge Decomposition Theorem
The set $\Omega^k(M)$ is $\binom{n}{k}$ dimensional over $C^\infty M$.

- There are $\binom{n}{k}$ ways to select $e^I$, and the set of $e^I$ are a pointwise basis for $\Omega^k(M)$.
- Functions in $C^\infty(M)$ denote a way to choose linear constants over each $T_pM$.
- In the case $k = 0$, we define $\Omega^0(M) = C^\infty(M)$. 

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The Hodge Decomposition Theorem
The Exterior Derivative

- The Exterior derivative $d_k$ maps $k$-forms to $k + 1$ forms.
- The Fundamental Theorem of Calculus exists on manifolds, and is stated in terms of $d$ (Stokes’ Theorem).

$$\int_M d\omega = \int_{\partial M} \omega$$
The Exterior Derivative

- When $k = 0$, a differential form is simply a smooth function $f : M \to \mathbb{R}$. This induces a linear map on tangent spaces:

$$df_p : T_p M \to T_{f(p)} \mathbb{R} = \mathbb{R}$$

- $df_p$ is 1-multilinear, alternating, and varies smoothly as a function of $p$. Thus, it is a 1-form!
The Exterior Derivative

- If \( \omega \) is a \( k \)-form and we write:

\[
\omega = \sum_{I=(i_1,\ldots,i_k)} f_I e^I
\]

then we define \( d\omega \) as:

\[
d\omega = \sum_{I=(i_1,\ldots,i_k)} df_i \wedge e^I.
\]

- We take the 0-form \( f_i \), make it into a 1-form, then “glue” it to \( e^I \) with the wedge product, making a \( k + 1 \) form.
The Exterior Derivative

- In practice, we need to know more algebra to compute $d\omega$.
- **Fun Facts:**
  1. If $f$ is a constant function, $df = 0$.
  2. $d$ is linear over $\mathbb{R}$.
  3. $d_{k+1}(d_k \omega) = 0$ for any $\omega \in \Omega^k(M)$.

\[ d \circ d = 0. \]
De Rham Cohomology

- $d_{k+1}(d_k \omega) = 0$ implies that $\ker d_{k+1} \subseteq \im d_k$.
- Define
  $$H^k_{dR}(M) = \frac{\ker d_{k+1}}{\im d_k}.$$  
- $H^k_{dR}(M)$ is a (possibly infinite-dimensional) vector space over $\mathbb{R}$, called the $k$-th de Rham Cohomology group of $M$.
- The dimension of $H^k_{dR}(M)$ roughly counts the number of $k$-dimensional holes in $M$. 

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The Hodge Decomposition Theorem
The Hodge Decomposition Theorem

**Theorem**

Let \((M, g)\) be a compact, Riemannian manifold. Then for each \(k = 1, \ldots, n\), the Hilbert space \(\Omega^k(M)\) of differential \(k\)-forms on \(M\) admits an orthonormal decomposition:

\[\Omega^k(M) = \text{im } d \oplus \text{im } \delta \oplus \ker \Delta_k.\]

Where \(d\) denotes the exterior derivative, \(\delta\) denotes the codifferential, and \(\Delta_k = d\delta + \delta d\) is the Hodge Laplacian on \(M\).

**Corollary**

\[H^k_{dR}(M) = \ker(\Delta_k)\]
The Codifferential

The codifferential $\delta$ is a map:

$$\delta : \Omega^k(M) \to \Omega^{k-1}(M)$$

defined by:

$$\delta \omega = (-1)^{n(n-k)+1} \ast d \ast \omega.$$

Fun Facts:

1. $\delta$ is linear over $\mathbb{R}$
2. $\delta \circ \delta = 0$

Definition

The $k$-th Hodge Laplacian $\Delta_k : \Omega^k(M) \to \Omega^k(M)$ is the mapping

$$\Delta_k = \delta d + d\delta.$$
The Hodge Star

Let $e_1, ..., e_n$ be an orthonormal basis for $T_p M$. Then we see that

$$V_g = e^1 \wedge ... \wedge e^n$$

which measures the $n$-dimensional volume such that

$$V_g(e_1, ..., e_n) = 1.$$

Suppose we take $e^{i_1} \wedge ... \wedge e^{i_k}$. We can determine $V_g$ if we also know $e^{j_1} \wedge ... \wedge e^{j_{n-k}}$ where the $j_\ell$’s are the indices that are complementary to $i_1, ..., i_k$. 

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The Hodge star $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ maps a $k$-form $\omega$ to $\star \omega \in \Omega^{n-k}(M)$ such that

$$\omega \wedge \star \omega = V_g.$$ 

This mapping is an isomorphism! so $\Omega^k(M) \cong \Omega^{n-k}(M)$. 

The Hodge Decomposition Theorem
The codifferential $\delta = (-1)^{n(n-k)+1} * d *$ can be explained by the following process:

1. Ignore $(-1)^{n(n-k)+1}$, it’s there for algebraic reasons.
2. Imagine a $k$-form $\omega$ as a way to measure $k$-dimensional subspace of a hypercube.
3. Instead of using $k$, measure the complementary $n - k$-dimensional volume of the hypercube given by $*\omega$.
4. Take the exterior derivative of $*\omega$ which gives an $n - (k - 1)$ dimensional volume.
5. Imagine the volume of the complementary $(k - 1)$-dimensional volume which is given by $*d * \omega$. 

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We can now define an inner product on $\Omega^k(M)$ by:

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \ast \eta$$

With respect to this inner product, $\delta$ is the adjoint to $d$. For all $\omega \in \Omega^{k-1}(M)$ and $\eta \in \Omega^k(M)$,

$$\langle d\omega, \eta \rangle = \langle \omega, \delta \eta \rangle$$
Theorem
Let \((M, g)\) be a compact, Riemannian manifold. Then for each \(k = 1, \ldots, n\), the Hilbert space \(\Omega^k(M)\) of differential \(k\)-forms on \(M\) admits an orthonormal decomposition:

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Corollary
\(H^k_{dR}(M) \cong \ker(\Delta_k)\)
Proof of the Corollary

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\[ H^k_{dR}(M) \cong \ker(\Delta_k) \]

Proof:

- We will show that the mapping

\[ \phi : \ker \Delta_k \to H^k_{dR}(M) = \frac{\ker d_k}{\text{im} \ d_{k-1}} \]

defined by:

\[ \phi(\omega) = [\omega] \]

is bijective.

- Decompose \( \omega \):

\[ \phi(\omega) = [\omega] = [\omega_d + \omega_\delta + \omega_\Delta] \]

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Claim: $[\omega_d] = [0]$

Proof: $\omega_d \in \text{im } d_{k-1}$

Claim: $\omega_\delta = 0$

Proof: Take the exterior derivative. We know $d\omega = 0$.

$$0 = d\omega = d\omega_d + d\omega_\delta + d\omega_\Delta.$$
Claim: $[\omega_d] = [0]$

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0 = d\omega = d \circ d\eta_1 + d \circ \delta\eta_2 + d\omega_\Delta.
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Proof of the Corollary

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\[
\|\omega_\delta\|^2 = 0
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Claim: \(\omega_\delta = 0\)

Proof: Take the exterior derivative. We know \(d\omega = 0\).

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\(\omega_\delta = 0\)
Proof of the Corollary

H^k_{dR}(M) \cong \ker(\Delta_k)

Proof:

We will show that the mapping

\[ \phi : \ker \Delta_k \to H^k_{dR}(M) = \frac{\ker \Delta_k}{\operatorname{im} d_{k-1}} \]

is bijective.

Decompose \( \omega \):

\[ \phi(\omega) = [\omega] + [\omega d] + [\omega \delta] + [\omega \Delta] \]

is bijective.

Defined by:

\[ \phi(\omega) = [\omega] \]
Proof of the Corollary

Corollary

\[ H^k_{dR}(M) \cong \ker(\Delta_k) \]

Proof:

- **Injective:** Let \( \omega \in \ker \Delta_k \) be such that

\[ \phi(\omega) = [0]. \]

We have:

\[ \phi(\omega) = [\omega] = [\omega_d] + [\omega_\delta] + [\omega_\Delta] \]
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\phi(\omega) = [\omega] = [0] + [0] + [\omega_\Delta].
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Corollary

\[ H^k_{dR}(M) \cong \ker(\Delta_k) \]

Proof:

- **Injective:** Let \( \omega \in \ker \Delta_k \) be such that \( \phi(\omega) = [0] \).

  We have:

  \[ \phi(\omega) = [\omega] = [\omega\Delta] = [0]. \]

  Therefore \( \omega\Delta = 0 \).
Corollary

\[ H^k_{dR}(M) \cong \ker(\Delta_k) \]

Proof:

- **Surjective:** Let \([\omega] \in H^k_{dR}(M)\). Then

  \[
  \phi(\omega_\Delta) = [0] + [0] + [\omega_\Delta].
  \]

  \[
  \phi(\omega_\Delta) = [\omega_d] + [\omega_\delta] + [\omega_\Delta]
  = [\omega]
  \]

Therefore \(\phi : \ker \Delta_k \to H^k_{dR}(M)\) is an isomorphism.
Proof of the Corollary

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- **Surjective**: Let \([\omega] \in H^k_{dR}(M)\). Then

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- **Surjective:** Let $[\omega] \in H^k_{dR}(M)$. Then

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\[
\phi(\omega_\Delta) = [\omega_d] + [\omega_\delta] + [\omega_\Delta] \\
= [\omega]
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Therefore $\phi : \ker \Delta_k \rightarrow H^k_{dR}(M)$ is an isomorphism.
In this talk, we have:

- Given an intuitive explanation of all of the terminology necessary to understand the Hodge Decomposition Theorem.
- Explained the relationship between the Hodge Laplacian and De Rham Cohomology.
- Proved that the kernel of the Hodge Laplacian is isomorphic to the De Rham Cohomology Groups of $\mathcal{M}$. 

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The Hodge Decomposition Theorem
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- Given an intuitive explanation of all of the terminology necessary to understand the Hodge Decomposition Theorem.
- Explained the relationship between the Hodge Laplacian and De Rham Cohomology.
Conclusion

In this talk, we have:

▶ Given an intuitive explanation of all of the terminology necessary to understand the Hodge Decomposition Theorem.
▶ Explained the relationship between the Hodge Laplacian and De Rham Cohomology.
▶ Proved that the kernel of the Hodge Laplacian is isomorphic to the De Rham Cohomology Groups of $M$. 
Thank You!

