Calculus Favorite: Stirling's Approximation, Approximately

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- In this short presentation, will try to give flavor of class – done with GMU honors and regular calculus 2; BC at magnet school as special appearance; in higher level TJ courses as an aside. Done interactively in class.

Start of class discussion

- **How big is $n!$?**
- **Say for $n = 1000$?**
- **We'll do this in computer algebra.**

Fun begins

From Mathematica: input is Factorial[1000] which outputs as

Crazy output

40238726007709377354370243392300398571937486421071463254379991042993851239862902059204420848696940480047

99886101971960586316668729948085589013238296699445909974245040870737599188236277271887325197795059509952

76120874975462497043601418278094646496291056393887437886487337119181045825783647849977012476632889835955

73543251318532395846307555740911426241747434934755342864657661166779739666882029120737914385371958824980

81268678383745597317461360853795345242215865932019280908782973084313928444032812315586110369768013573042

16168747609675871348312025478589320767169132448426236131412508780208000261683151027341827977704784635868

17016436502415369139828126481021309276124489635992870511496497541990934222156683257208082133318611681155

36158365469840467089756029009505376164758477284218896796462449451607653534081989013854424879849599533191

01723355556602139450399736280750137837615307127761926849034352625200015888535147331611702103968175921510

90778801939317811419454525722386554146106289218796022383897147608850627686296714667469756291123408243920

81601537808898939645182632436716167621791689097799119037540312746222899880051954444142820121873617459926

42956581746628302955570299024324153181617210465832036786906117260158783520751516284225540265170483304226

14397428693306169089796848259012545832716822645806652676995865268227280707578139185817888965220816434834

48259932660433676601769996128318607883861502794659551311565520360939881806121385586003014356945272242063

44631797460594682573103790084024432438465657245014402821885252470935190620929023136493273497565513958720

55965422874977401141334696271542284586237738753823048386568897646192738381490014076731044664025989949022

Crazy output – Continued

22217659043399018860185665264850617997023561938970178600408118897299183110211712298459016419210688843871
21855646124960798722908519296819372388642614839657382291123125024186649353143970137428531926649875337218
94069428143411852015801412334482801505139969429015348307764456909907315243327828826986460278986432113908
35062170950025973898635542771967428222487575867657523442202075736305694988250879689281627538488633969099
59826280956121450994871701244516461260379029309120889086942028510640182154399457156805941872748998094254
74217358240106367740459574178516082923013535808184009699637252423056085590370062427124341690900415369010
5933983835777939410970027753472000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
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Only off by about 2×10^{131} – not too bad?!

Further exploration begins

Soon get discussion to ask about $n!$ and its definition as repeated multiplication

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Consider the summation idea using ln(n!)

$$\ln(n!) = \sum_{k=1}^{n} \ln(k)$$

and now compare sum to integral using left and right endpoints (and soon midpoint).

Graphical view

Here is a graph, using 10 instead of 1000 so we can see things.

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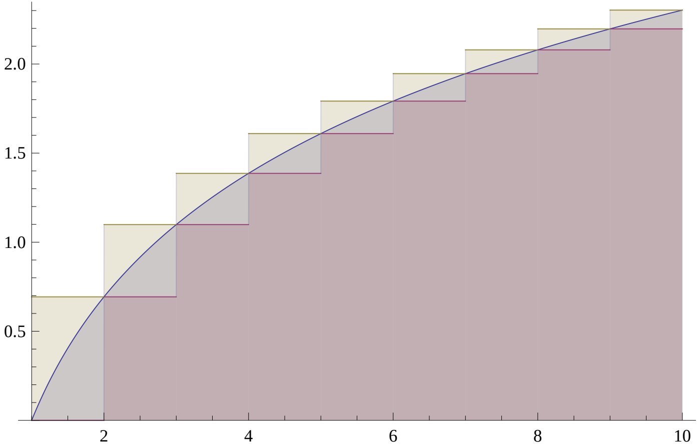

More graphics

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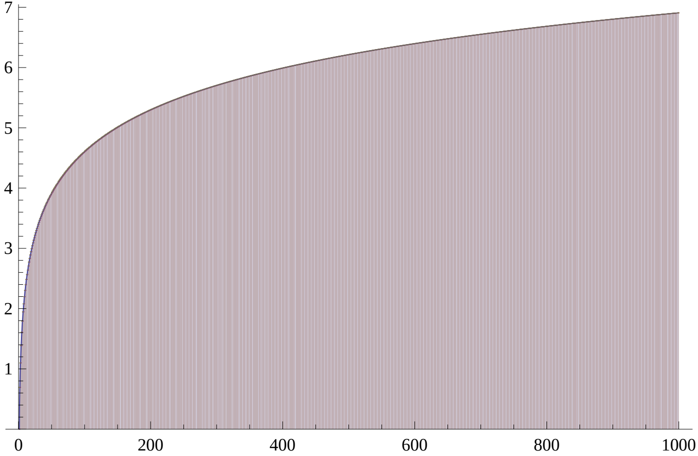

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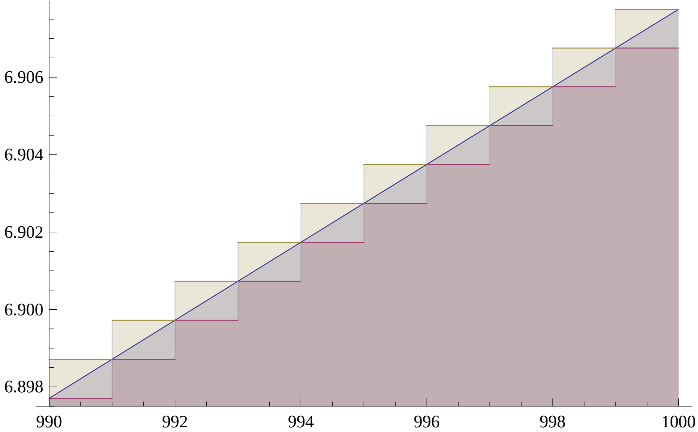

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Graph increases, so left endpoint sum is lower, right endpoint is higher. This yields some estimates:

$$\ln(n!) - \ln(n) = \sum_{k=1}^{n-1} \ln(k) < \int_1^n \ln(x)\,dx < \sum_{k=2}^{n} \ln(k) = \ln(n!)$$

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This yields the initial estimates:

$$(n/\mathrm{e})^n\,\mathrm{e} < n! < (n/\mathrm{e})^n\, n\,\mathrm{e}$$

Trapezoid approximation

Using the trapezoid approximation rather than endpoints does a better job (average of left and right)

$$\int_1^n \ln x \, dx \approx \sum_{k=2}^{n} \left(\frac{\ln(k-1) + \ln(k)}{2}\right) = \ln(n!) - \frac{1}{2} \ln(n)$$

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Correct except numerical factor: e vs. $\sqrt{2\pi}$.

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Numerical values are as follows

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Full expansion can be had with some extra effort (Euler-Maclaurin formula).

Fancy script writing on the frosting on the cake

From graphs it is clear most of the error is in the early terms. Using the discrete sum for a few steps and then using the integral cuts the numerical discrepancy. Here are some easy first few steps:

$$\begin{aligned} ((\mathrm{e} * 2)/3)^{\frac{3}{2}} &\approx 2.439522535141459 3 \\ 2 * (2 * \mathrm{e}/5)^{\frac{5}{2}} &\approx 2.465563423812403 \\ 2 * 3 * (2 * \mathrm{e}/7)^{\frac{7}{2}} &\approx 2.477101383175650 \end{aligned}$$

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

and Stirling says $n!$ is much larger than the exponential term $(x-a)^n$, so it boils down to the growth rate (in n) of the derivatives.

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For exponentials and basic trig (sine and cosine) the factorial wins and the radius is infinite, while for fractional and negative powers, and their integrals or derivatives, there is a balance and a finite radius.

The exact value for the leading constant term

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Came in the context of probability for repeated Bernoulli trials, fair coin. $2n$ tries, exactly n of heads, tails. Need the central binomial coefficient:

$$\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$$

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The full asymptotic expansion can be done by Laplace's method, starting from the formula $n! = \int_0^\infty t^n \, \mathrm{e}^{-t} \, dt$.

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Thank you for your attention and I welcome your comments and/or questions.