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Partial Differential Equations in the 20th Century*

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1. INTRODUCTION

The study of partial differential equations (PDE's) started in the 18th century in the work of Euler, d'Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. The analysis of physical models has remained to the present day one of the fundamental concerns of the development of PDE's. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDE's also became an essential tool in other branches of mathematics.

This duality of viewpoints has been central to the study of PDE's through the 19th and 20th century. On the one side one always has the relationship to models in physics, engineering and other applied disciplines. On the other side there are the potential applications—which have often turned out to be quite revolutionary—of PDE's as an instrument in the development of other branches of mathematics. This dual perspective was clearly stated for the first time by H. Poincaré [Po1] in his prophetic paper in 1890. Poincaré emphasized that a wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance—un "air de famille" in Poincaré's words—and should be treated by common methods. He also explained the interest in having completely rigorous proofs, despite the fact that the models are only an approximation of the physical reality. First, the mathematician desires to carry through his research in a precise and convincing form. Second, the resulting theory is applied as a tool in the study of major mathematical areas, such as the Riemann analysis of Abelian functions.

In the same paper there is also a prophetic insight that quite different equations of mathematical physics will play a significant role within mathematics itself. This has indeed characterized the basic role of PDE, throughout the whole 20th century as the major bridge between central issues of applied mathematics and physical sciences on the one hand and the central development of mathematical ideas in active areas of pure mathematics. Let us now summarize some areas in mathematics which have had a decisive interaction with PDE's.

The first great example is Riemann's application of a potential theoretic argument, the Dirichlet principle and its uses, in developing the general theory of analytic functions of a complex variable and the related theory of Riemann surfaces. Generalizing the latter was the extension, beginning with Hodge theory, of comparable tools in the study of *algebraic geometry* in several variables. It led to such developments as the Riemann–Roch theorem and the Atiyah–Singer index theorem.

The next major example is *differential geometry*, especially in its global aspects. Topics in differential geometry, such as minimal surfaces and imbedding problems giving rise to the Monge–Ampère equations, have stimulated the analysis of PDE's, especially nonlinear equations. On the other hand, the creation of powerful analytical tools in PDE's (a priori estimates) have made it possible to answer fundamental open questions in differential geometry. This interplay has revolutionalized the field of differential geometry in the last decades of the 20th century.

On the other hand the theory of systems of first order partial differential equations has been in a significant interaction with Lie theory in the original work of S. Lie, starting in the 1870's, and E. Cartan beginning in the 1890's. The theory of exterior differential forms has played an increasingly important role since their introduction and use by E. Cartan, and the introduction of sheaf theory by Leray in 1945 has led to a dramatic union of ideas and techniques from manifold theory, algebraic and differential topology, algebraic geometry, homological algebra and microlocal analysis (see the book of Kashiwara and Schapira [Ka-Sc]).

The need for a rigorous treatment of solutions of PDE's and their boundary value problems (= BVP's), was a strong motivation in the development of basic tools in *real analysis* and *functional analysis* since the beginning of the 20th century. This perspective on the development of functional analysis was clearly laid out by J. Dieudonné [Di] in his history of functional analysis. Starting in the 1950's and 60's the systematic study of linear PDE's and their BVP's gave rise to a tremendous extension of techniques in *Fourier analysis*. The theory of singular integral operators, which started in the 1930's in connection with PDE's, has become, through the Calderon–Zygmund theory and its extensions, one of the central themes in harmonic analysis. At the same time the applications of Fourier analysis to PDE's through such tools as pseudo-differential operators and Fourier integral operators gave an enormous extension of the theory of linear PDE's.

Another example is the interplay between PDE's and *topology*. It arose initially in the 1920's and 30's from such goals as the desire to find *global* solutions for nonlinear PDE's, especially those arising in fluid mechanics, as in the work of Leray. Examples, in the 1920's, are the variational theories of M. Morse and Ljusternik–Schnirelman, and in the 1930's, the Leray–Schauder degree in infinite dimensional spaces as an extension of the classical Brouwer degree. After 1960 the introduction of a variational viewpoint in the study of differential topology gave rise to such important results as Bott's periodicity theorem, and Smale's proof of the Poincaré conjecture for dimension $\geqslant 5$. More recently, the analysis of the Yang–Mills PDE has given rise to spectacular progress in low dimensional topology.

Another extremely important connection involving PDE's as a bridge between central mathematical issues and practical applications takes place in the field of *probabilistic models*, the so-called stochastic processes. It arose initially from the study of Brownian motion by Wiener (in the 20's and early 30's) and was extended by Ito, Levy, Kolmogorov and others, to a general theory of stochastic differential equations. More recently it has given rise to the Malliavin program using infinite dimensional Sobolev spaces. This theory is closely connected to diffusion PDE's, such as the heat equation. Stochastic differential equations are now the principal mathematical tool for the highly active field of option pricing in finance.

Another striking example is the relationship between *algebraic geometry* and the soliton theory for the Korteweg–DeVries PDE. This equation was introduced in 1896 as a model for water waves and has been decisively revived by M. Kruskal and his collaborators in the 1960's; see Section 20.

The study of the asymptotic behavior of solutions of nonlinear equations of evolution, particularly those governing fluid flows and gas dynamics, has been an important arena for the interaction between PDE's and current themes in *chaos theory*. This is one of the possible approaches to the central problem of *turbulence*—one of the major open problems in the physical sciences.

There are many other areas of contemporary research in mathematics in which PDE's play an essential role. These include infinite dimensional group representations, constructive quantum field theory, homogeneous spaces and mathematical physics.

Finally, and this may be the most important from the practical point of view, computations of solutions of PDE's is the major concern in *scientific computing*. This was already emphasized by Poincaré in 1890, though the practicality of the techniques available in his time was extremely limited as Poincaré himself remarked. Today with the advent of high-speed supercomputers, computation has become a central tool of scientific progress.

2. MODELS OF PDE'S IN THE 18TH AND 19TH CENTURY

PDE arose in the context of the development of models in the physics of continuous media, e.g. vibrating strings, elasticity, the Newtonian gravitational field of extended matter, electrostatics, fluid flows, and later by the theories of heat conduction, electricity and magnetism. In addition, problems in differential geometry gave rise to nonlinear PDE's such as the Monge–Ampère equation and the minimal surface equations. The classical calculus of variations in the form of the Euler–Lagrange principle gave rise to PDE's and the Hamilton-Jacobi theory, which had arisen in mechanics, stimulated the analysis of first order PDE's.

During the 18th century, the foundations of the theory of a single first order PDE and its reduction to a system of ODE's was carried through in a reasonably mature form. The classical PDE's which serve as paradigms for the later development also appeared first in the 18th and early 19th century.

The one dimensional *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

was introduced and analyzed by d'Alembert in 1752 as a model of a vibrating string. His work was extended by Euler (1759) and later by D. Bernoulli (1762) to 2 and 3 dimensional wave equations

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \text{where} \quad \Delta u = \sum_i \frac{\partial^2}{\partial x_i^2}$$

in the study of acoustic waves ($\sum$ refers to the summation over the corresponding indices).

The Laplace equation

$$\Delta u = 0$$

was first studied by Laplace in his work on gravitational potential fields around 1780. The *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u$$

was introduced by Fourier in his celebrated memoir "Théorie analytique de la chaleur" (1810–1822).

Thus, the three major examples of second-order PDE's—hyperbolic, elliptic and parabolic—had been introduced by the first decade of the 19th century, though their central role in the classification of PDE's, and related boundary value problems, were not clearly formulated until later in the century.

Besides the three classical examples, a profusion of equations, associated with major physical phenomena, appeared in the period between 1750 and 1900:

- The Euler equation of incompressible fluid flows, 1755.
- The minimal surface equation by Lagrange in 1760 (the first major application of the Euler–Lagrange principle in PDE's).
- The Monge–Ampère equation by Monge in 1775.
- The Laplace and Poisson equations, as applied to electric and magnetic problems, starting with Poisson in 1813, the book by Green in 1828 and Gauss in 1839.

- The Navier Stokes equations for fluid flows in 1822–1827 by Navier, followed by Poisson (1831) and Stokes (1845).
- Linear elasticity, Navier (1821) and Cauchy (1822).
- Maxwell's equation in electromagnetic theory in 1864.
- The Helmholtz equation and the eigenvalue problem for the Laplace operator in connection with acoustics in 1860.
- The Plateau problem (in the 1840's) as a model for soap bubbles.
- The Korteweg–De Vries equation (1896) as a model for solitary water waves.

A central connection between PDE and the mainstream of mathematical development in the 19th century arose from the role of PDE in the theory of analytic functions of a complex variable. Cauchy had observed in 1827 that two smooth real functions u, v of two real variables x, y are the real and imaginary parts of a single analytic complex function of the complex variable $z = x + iy$ if they satisfy the Cauchy–Riemann system of first order equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

From the later point of view of Riemann (1851) this became the central defining feature of analytic functions. From this point of view, Riemann studied the properties of analytic functions by investigating harmonic functions in the plane.

3. METHODS OF CALCULATING SOLUTIONS IN THE 19TH CENTURY

During the 19th century a number of important *methods* were introduced to find solutions of PDE's satisfying appropriate auxiliary boundary condition:

(A) *Method of separation of variables and superposition of solutions of linear equations.* This method was introduced by d'Alembert (1747) and Euler (1748) for the wave equation. Similar ideas were used by Laplace (1782) and Legendre (1782) for the Laplace's equation (involving the study of spherical harmonics) and by Fourier (1811–1824) for the heat equation.

Rigorous justification for the summation of infinite series of solutions was only loosely present at the beginning because of a lack of efficient criteria for the convergence of functions (this was instituted only after the 1870's as part of the rigorization of analysis). This question led to extremely important developments in analysis and mathematical physics, in particular Fourier series and integrals.

(B) The interplay between the study of 2-dimensional real harmonic functions and analytic functions of a single complex variable which originated in the work of Riemann (1851) was extensively developed by C. Neumann, H. A. Schwarz, and E. B. Christoffel around 1870.

(C) The method of Green's functions was introduced in 1835 for the Laplace equation. It consists of studying *special singular solutions* of the Laplace equation. These solutions are then used to represent solutions satisfying general boundary conditions or with arbitrary inhomogenous terms.

(D) An extremely important principle was discovered by G. Green in 1833 for the Laplace equation. He observed that a solution of the equation

$$\Delta u = 0 \qquad \text{in a domain} \quad \Omega \subset \mathbb{R}^3$$

which assumes a given boundary value, $u = \varphi$ on the boundary $\partial\Omega$ of Ω (later called the Dirichlet problem), minimizes the integral

$$\int_\Omega \sum_{i=1}^{3} \left(\frac{\partial v}{\partial x_i}\right)^2$$

among all functions v such that $v = \varphi$ on $\partial\Omega$. If there is a minimizer u which is smooth, then it is a harmonic function. Related arguments were carried out independently by Gauss. Their work was followed by W. Thomson (= Lord Kelvin) in 1847 and by Riemann in his thesis in 1851 where he named this approach the *Dirichlet principle*.

(E) Though *power series methods* had been used by Euler, d'Alembert, Laplace and others, to obtain particular solutions of PDE's, a systematic use of power series, especially in connection with the initial value problem for nonlinear PDE's, was started by Cauchy in 1840. This began work on existence theory, even when explicit solutions are not available. The method of Cauchy, known as the method of majorants to obtain real analytic solutions, i.e., expandable in convergent power series, was extended in 1875 by Sophie Kowalewsky to general systems and simplified by Goursat in 1898.

A general survey of the development of PDE's in the 18th and 19th century is given in volume 2 of Kline's book [Kli]. The treatment of the history of rational mechanics and PDE's in the 18th century is based on the publications of C. Truesdell as in his very interesting paper [Tru].

4. DEVELOPMENTS OF RIGOROUS THEORIES OF SOLVABILITY IN THE LAST DECADES OF THE 19TH CENTURY

Up to about 1870 the study of PDE was mainly concerned with heuristic methods for finding solutions of boundary value problems for P.D.E.'s, as well as explicit solutions for particular problems. Under the influence of the rigorization program for analysis led by Weierstrass around 1870, systematic attention began to be paid to finding rigorous proofs of basic existence results. The most conspicuous case was the Dirichlet problem introduced by Riemann in 1851 which asks for the solution of the equation

$$\Delta u = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^2$$

which satisfies the boundary condition

$$u = \varphi \qquad \text{on } \partial\Omega.$$

Riemann had reduced the solvability of this problem to the existence of a smooth minimizing function for the Dirichlet integral

$$E(v) = \int_\Omega \sum_i \left(\frac{\partial v}{\partial x_i}\right)^2$$

over the class of functions satisfying the condition $v = \varphi$ on $\partial\Omega$. Though he had given an electrostatic model for the Dirichlet principle, he had not proved the existence of a minimizer by any mathematically satisfactory method, as was pointed out by Weierstrass and his school.

The criticism of Riemann's argument was in two directions. First, for functionals apparently similar to the Dirichlet integral it was shown that no minimizer exists. On the other hand, F. Prym, in 1871, gave an example of a continuous boundary datum defined on the circle for which no extension in the disc has finite energy. Thus, the legitimacy of Riemann's Dirichlet principle as a tool for proving existence of harmonic functions was put in serious doubt for several decades. This program was reinstated as a major theme of mathematical research by Hilbert in 1900 and gave rise to an extensive development of methods in this domain (see Section 6).

As a result of the attention drawn by Riemann to the significance of the study of harmonic functions (potential theory) in geometric function theory, other approaches to the existence of a solution for the Dirichlet problem were developed in the last three decades of the 19th century. The alternating method of H. A. Schwarz (around 1870) consists of splitting the domain Ω into two pieces and then solving in alternation the Dirichlet problem on each of these domains. In 1877 C. Neumann introduced the method of *integral equations* for the Dirichlet problem in a convex domain via the representation of possible solutions by double layer potentials. This approach was developed more systematically during the next decade by Poincaré (see Section 5) and later by Fredholm and Hilbert (see Section 6 and [Kli, Vol. 3]).

5. THE PERIOD 1890-1900: THE BEGINNING OF MODERN PDE AND THE WORK OF POINCARÉ

The main contributions of Poincaré to the theory of PDE's are the following:

(a) In 1890 Poincaré [Po1] gave the first complete proof, in rather general domains, of the existence and uniqueness of a solution of the Laplace equation for any continuous Dirichlet boundary condition. He introduced the so-called *balayage method*; this iterative method relies on solving the Dirichlet problem on balls in the domain and makes extensive use of the maximum principle and Harnack's inequality for harmonic functions. A systematic exposition of this method was given in his lectures of 1894–95 at the Sorbonne and published in [Po4]. Together with books of Harnack and Korn this is the origin of the extensive development of potential theory in the following decades. The interested reader will find a detailed summary of potential theory up to 1918 in the Encyklopädia article [Li2] of Lichtenstein. We note that, as pointed out in Section 19, the maximum principle for second order elliptic and parabolic equations has played a central role throughout the 20th century.

(b) In a fundamental paper of 1894, Poincaré [Po2] established the existence of an infinite sequence of *eigenvalues* and corresponding *eigenfunctions* for the Laplace operator under the Dirichlet boundary condition. (For the first eigenvalue this was done by H. A. Schwarz in 1885 and for the second eigenvalue by E. Picard in 1893.) This key result is the beginning of spectral theory which has been one the major themes of functional analysis and its role in theoretical physics and differential geometry during the 20th century; for more details, see Dieudonné's history of functional analysis [Di] and Section 18.

(c) Picard and his school, beginning in the early 1880's, applied the method of *successive approximation* to obtain solutions of nonlinear problems which were mild perturbations of uniquely solvable linear problems. Using this method, Poincaré [Po3] proved in 1898 the existence of a solution of the nonlinear equation

$$\Delta u = e^u$$

which arises in the theory of Fuchsian functions. Motivated by this problem as well as many nonlinear problems in mathematical physics, Poincaré formulated the principle of the *continuity method*. This attempts to obtain solutions of nonlinear equations by embedding them in a one-parameter family of problems, starting with a simple problem and attempting to extend solvability by a step-by-step change in the parameter. This became a major tool in the bifurcation theory of A. M. Lyapunov, E. Schmidt and others, as well in the existence theory for nonlinear elliptic equations as developed by S. Bernstein, J. Leray and J. Schauder (see Sections 7, 9 and 21).

6. THE HILBERT PROGRAMS

In his celebrated address to the international mathematical Congress in Paris in 1900, Hilbert presented 23 problems (the so-called Hilbert problems), two of which are concerned with the theory of nonlinear elliptic PDE's. Though initially restricted to a variational setting, Hilbert's problems 19 and 20 set the broad agenda for this area in the 20th century.

Problem 19 addresses the theme of *regularity* of solutions (specifically in this case analyticity of solutions). Problem 20 concerns the question of *existence* of solutions of boundary value problems and, in particular, the existence of solutions which minimize variational principles.

In connection with Problem 20, Hilbert revived the interest in Riemann's approach to the Dirichlet principle. The methods originally proposed by Hilbert during the period 1900–1905 for the Dirichlet principle are complex and difficult to follow, but gave rise to an extensive attack by numerous authors, e.g. B. Levi, H. Lebesgue, G. Fubini, S. Zaremba, L. Tonelli and R. Courant, which was very fruitful in creating new tools, e.g. see [Li2]. The original suggestion of Hilbert [Hi1] was to take a minimizing sequence for the Dirichlet integral and to prove that an appropriate modified sequence converges uniformly to a minimizer. A variant of this approach was carried through a few years later by S. Zaremba using a "mollified" form of the original minimizing sequence. Another version was presented by R. Courant (e.g. see his book [Co]). These arguments, following Hilbert's original suggestion, rely upon a compactness argument in the uniform topology, namely Ascoli's theorem. One must recall that in 1900 the theory of L^p spaces in terms of the Lebesgue integral, and their completeness had not yet been formulated. It was B. Levi [LB] who first observed in 1906 that a general minimizing sequence for the Dirichlet integral is a Cauchy sequence in the Dirichlet norm, and therefore converges in an appropriate completion space (with respect to the Dirichlet norm) to a generalized function. With this observation he began the essential study of function spaces associated with the direct method of the Calculus of Variations; they are now called the Sobolev spaces; see Section 12.

A solution of Problem 19 was carried through for general second order nonlinear elliptic equations in 2 dimensions by S. Bernstein beginning in 1904 (see Section 7). His methods gave rise to essential techniques of establishing a priori estimates for solutions and their derivatives, in particular, using the linearization of nonlinear equations in a neighborhood of a solution. (For a detailed discussion of developments arising from Hilbert problems 19 and 20, see the articles by J. Serrin and G. Stampacchia in the volume "Mathematical developments arising from Hilbert problems" published by the AMS in 1976.)

Following up on the results of Poincaré and J. Fredholm (1903), Hilbert, in his papers on linear integral equations [Hi2], formulated a general program for establishing the existence and completeness of eigenfunctions for linear self adjoint integral operators and applying these results to PDE's.

7. S. BERNSTEIN AND THE BEGINNING OF A PRIORI ESTIMATES

In his papers [Be2], beginning in 1906, on the solvability of the Dirichlet problem for nonlinear elliptic equations, S. Bernstein observed that in order to carry through the continuity method, it is essential to establish that the size of the interval in the parameter in the step-by-step argument does not shrink to zero as one proceeds. This fact will follow if one shows that the solutions obtained via this continuation process lie in a compact subset of an appropriate function space. Such a property is usually established by showing that prospective solutions and their derivatives of various orders satisfy *a priori bounds*. In the case that Bernstein studied—second order nonlinear elliptic equations in the plane—he developed the first systematic method for such estimates. These techniques were extensively sharpened over many decades; see Sections 8, 16, 19 and 23.

As a simple illustration of the possibilities and the difficulties of this approach let us consider two simple examples of a semilinear elliptic equation:

$$(a)\quad \begin{cases} -\Delta u + u^3 = f(x) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$$(b)\quad \begin{cases} -\Delta u - u^2 = f(x) & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The continuity method amounts to introducing a parameter $t \in [0, 1]$ connecting the given problem to a simpler equation, usually linear. For example, in the two cases above the equations become

$$(a_t) \quad \begin{cases} -\Delta u + tu^3 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

$$(b_t) \quad \begin{cases} -\Delta u - tu^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To show the solvability for $t = 1$ one tries to prove that the set of parameter values of t in $[0, 1]$ for which the problem (a_t) or (b_t) is solvable is both open and closed.

If for a given parameter value t_0, u_0 is the corresponding solution of (a_{t_0}) for example, the solvability of the problem for t near t_0 in a given functional space X would follow from the implicit function theorem once the *linearized* problem in the new variable v is uniquely solvable. For example, for (a_t), the linearized problem is

$$(L_{t_0}) \quad \begin{cases} -\Delta v + 3t_0 u_0^2 v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $v \in X$.

The coefficients of the linearized problem depend on u_0 which is an element of the function space X. This fact became a major impetus in the fine study of linear equations with coefficients in various function spaces (see Section 8). The choice of the function space X is not arbitrary but also depends on the other step, i.e., whether the set of parameter values for which solvability holds, is closed in $[0, 1]$.

The proof that the set of values of $t \in [0, 1]$ for which (a_t) or (b_t) has a solution u_t is closed, relies on estimates which hold for all possible solutions. Usually, one proves that (u_t) lies in a compact set of the function space X. For a sequence $t_k \to t$ we can therefore extract a convergent subsequence u_{t_k} in X which converges to a solution u_t of (a_t).

Thus, we have opposite requirements on X. For Step 1 to hold it is useful to have as much regularity as possible for the functions in X. For Step 2 and the a priori estimate it is preferable to require as little as possible. The successful completion of the argument requires a choice of X which balances these requirements.

For example, in the cases we have listed above, in (a_t), the most natural space is $X = C^2(\bar{\Omega})$. But as was observed at the beginning of this century the linear equations $\Delta u = g$ does not necessarily have a solution in C^2 for $g \in C^0$. Thus, the invertibility of linear elliptic operators in function spaces became a matter of serious concern. The space which is useful in place of

C^0 is the space $C^{0,\alpha}$ of functions g satisfying a *Hölder condition* with exponent $\alpha \in (0, 1)$

$$|g(x)-g(y)| \leqslant C\,|x-y|^{\alpha}.$$

For further details see Section 8.

An important consideration in carrying both Step 1 and Step 2 has been the application of the celebrated maximum principle for a linear elliptic equation of second order (see Section 19). In the maximum principle, for linear operators, the sign of the coefficient of the zero order term plays a decisive role. For example, the positivity of this coefficient in L_{t_0} insures that the maximum principle applies and the linear problem is uniquely solvable. The continuity method can be carried through for problem (a) and yields a solution for every given f. By contrast this method cannot be applied to problem (b) because lack of control of the sign of the coefficient of v. Indeed, problem (b) can be shown to have solutions only for restricted choices of f.

In applying his methods to existence proofs, Bernstein restricted himself to cases where the perturbed problem can be solved by successive approximation. Thirty years later, J. Leray and J. Schauder combined the techniques of a priori estimates à la Bernstein with concepts drawn from topology, e.g. the degree of mappings. This considerably enlarged the class of application by removing the restriction of unique solvability of the linearized problem; see Section 9.

S. Bernstein [Be1], in 1904, gave a positive solution of Hilbert's Problem 19. He proved that a C^3 solution of a general fully nonlinear second order elliptic equation (the precise meaning of these terms is given in Section 23) in the plane,

$$F(x, y, u, Du, D^2u)=0$$

is analytic whenever F is analytic. To carry through this proof, S. Bernstein established estimates for derivatives of solutions given in the form of power series. At the end of his argument he observed that such methods could be used to obtain a positive solution of Hilbert's Problem 20 concerning the existence of solutions of the Dirichlet problem. In subsequent papers over several decades, Bernstein developed this program and established the first systematic method to obtain existence via a priori estimates.

Schauder [Sca2] returned to this problem in 1934 and disconnected the topics of analyticity and existence. He observed that the appropriate estimates for the existence in the quasilinear case are $C^{2,\alpha}$ estimates. It is these estimates which were applied by Leray–Schauder (see Section 9).

By contrast, the initial regularity in which existence is established via the direct method of the calculus of variations is much weaker than C^3: the solution belongs to some Sobolev space $W^{1,p}$ (see Section 12) and the question arises whether such weak solutions are smooth. This problem was successfully solved by C. Morrey [Mor2] in 1943 in 2 dimensions and the general case was finally settled by E. DeGiorgi and J. Nash in 1957 (see Section 19).

8. SOLVABILITY OF SECOND ORDER LINEAR ELLIPTIC EQUATIONS

Following the work of Neumann and the development of a systematic theory of integral equations by Poincaré, Fredholm, Hilbert and others, there was a general attack on studying the solutions of second order linear elliptic equations obtained by integral representation. The construction of elementary solutions and Green's functions for general higher order linear elliptic operators was carried through in the analytic case by E. E. Levi (1907) [Le]. The parametrix method was also applied by Hilbert and his school in the study of particular boundary value problems.

An important technical tool in this theory was the introduction of Hölder conditions by O. Hölder in 1882 in his book [Hol] on potential theory. The study of single and double layer potentials with densities lying in Hölder spaces became the subject of intensive investigations through the works of Lyapunov (1898), A. Korn [Kor] (1907), in connection with the equations of elasticity, L. Lichtenstein starting in 1912 (see the scholarly exposition [Li2] in the volume on potential theory in the Encyklopädia der Math. Wiss.) and P. Levy (1920).

Following the treatment of harmonic functions by Kellogg in his book [Ke] on potential theory (1929), Schauder [Sca2] and, shortly afterwards CaccioPoli [Ca1], applied these techniques to obtain a priori estimates in $C^{2,\alpha}$ spaces for the solutions of the Dirichlet problem for linear elliptic equations of second order with $C^{0,\alpha}$ coefficients. More specifically if one postulates *a priori* the existence of a $C^{2,\alpha}$ solution for the equation

$$\begin{cases} Au = \sum_{ij} a_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i a_i(x) \dfrac{\partial u}{\partial x_i} + a_0(x)\, u = f(x) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

then there is a constant C, depending only on the domain Ω and the coefficients, such that

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C(\|f\|_{C^{0,\alpha}(\bar{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\partial\Omega)} + \|u\|_{C^0(\bar{\Omega})}).$$

Here, the Hölder norms are given by

$$\|v\|_{C^{0,\alpha}} = \operatorname*{Sup}_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\alpha}} + \operatorname*{Sup}_{x} |v(x)|$$

and

$$\|v\|_{C^{2,\alpha}} = \sum_{i,j} \left\| \frac{\partial^2 v}{\partial x_i \, \partial x_j} \right\|_{C^{0,\alpha}} + \sum_{i} \left\| \frac{\partial v}{\partial x_i} \right\|_{C^{0,\alpha}} + \|v\|_{C^{0,\alpha}}.$$

In his paper [Sca2], Schauder explicitly carries through the program of establishing *existence* results for these linear problems combining the method of a priori estimates with the theory of F. Riesz for linear compact operators in Banach spaces. This became a major bridge between functional analysis and the theory of PDE. It is this viewpoint of Schauder, combined with algebraic topology, which was carried over to nonlinear equations by Leray–Schauder; see Section 9.

9. LERAY–SCHAUDER THEORY

In the work of S. Bernstein (see Section 7) existence results, obtained by continuation techniques, relied upon uniqueness conditions for the solutions of the linearized problem. This restricted considerably the class of equations which could be treated by that method. The contribution of Leray–Schauder in their famous paper [L-S] of 1934 was to get rid of the uniqueness condition and rely exclusively upon a priori estimates and topological methods.

The principal tool which they applied was a major advance in *nonlinear functional analysis*, the extension to infinite dimensional spaces of the degree of mappings. Following earlier partial results of Birkhoff–Kellogg on extensions of the Brouwer fixed point theorem to infinite dimensions, Schauder [Sca1] in 1930 had established the fundamental fixed point theorem asserting that a compact mapping from a ball into itself has a fixed point (a mapping is said to be compact if it is continuous and has relatively compact image). In 1929–32 Schauder generalized the Brouwer principle of invariance of domains for maps of the form $(I - C)$ where C is compact and I denotes the identity map. In 1934 Leray and Schauder [L-S] extended the Brouwer degree of mappings to the class of maps of the form $(I - C)$ and applied this theory, combined with a priori estimates to obtain existence theorems for quasilinear second order equations in the plane. This generated a vast new program to obtain further existence results by establishing appropriate a priori estimates.

The heart of this method lies in the most important property of degree. The degree, $\deg(I-C, G, p)$, is an algebraic count of the number of solutions of the equation

$$(I-C)\,u = p, \qquad u \in G$$

where G is a bounded open set in a Banach space X. This degree is only defined if there is no solution of that equation on the boundary of G. The degree is invariant under continuous deformation C_t of the mapping, provided that it remains defined during a continuous compact deformation, i.e., no solution of the equation appears on the boundary during the deformation.

To apply this principle, for example when G is a ball, one must show that no solution appears on the boundary of the ball. In practice, one shows by a priori estimates, that all solutions lie inside a fixed ball. One constructs the deformation C_t to connect the given problem $C = C_1$ with a simple problem for which the degree can be computed easily, e.g. $C_0 = 0$. The proof of the necessary a priori estimates has often posed difficult problems, some of which have been resolved only after decades of intensive work. The most striking example is the Monge–Ampère equation

$$\det(D^2 u) = f(x)$$

for which the estimates were completed only in the 1980's (see Section 23).

10. HADAMARD AND THE CLASSIFICATION OF PDE'S AND THEIR BOUNDARY VALUE PROBLEMS

One knows, in the study of classical PDE's (Laplace, heat, wave equations), that there are very specific kinds of boundary conditions usually associated with each of these equations. For the Laplace equation one has the Dirichlet condition ($u = \varphi$ on $\partial\Omega$) or the Neumann condition (where one prescribes the normal derivative $\partial u/\partial n$ on $\partial\Omega$). For the heat equation the classical boundary condition is to prescribe the initial value of the solution (and in the case of a bounded domain, the Dirichlet condition on the boundary of the domain for positive time). In the case of the wave equation, the most classical boundary value problem is the Cauchy problem which prescribes both the initial position and the initial velocity (at $t = 0$).

The ground for telling whether a boundary condition is appropriate for a given PDE is often physically obscure. It has to be clarified by a fundamental mathematical insight. The basic principle for distinguishing "legitimate" or well-posed problems was stated clearly by Hadamard in 1923 in his book [Ha] on the Cauchy problem in the following terms: the solution should exist on a prescribed domain for all suitable boundary data, should be uniquely determined by such data and be "stable" in terms of appropriate norms.

Thus, for example, for the Cauchy problem, the theorem of Cauchy-Kowalevska, proved in the 19th century for equations with analytic data, establishes the existence of solutions in power series for equations which are not characteristic with respect to the initial surface. This includes the Laplace equation for example. However, in this case, the domain of existence of the solution varies drastically with the data and the solutions are highly unstable with respect to the boundary data. Thus, this problem is ill-posed in the Hadamard sense.

Hadamard also proposed to find general classes of equations having distinctive properties for their solutions in terms of the characteristic polynomials. This is the polynomial obtained by replacing each partial derivative $\partial/\partial x_j$ by the algebraic variable ξ_j and keeping the top order part in each variable. We thus obtain, in particular, basic classes of second order operators, called elliptic, hyperbolic and parabolic which are, respectively, generalizations of the Laplace operator, the wave operator and the heat operator. The elliptic operators are defined by quadratic polynomials which vanish only at $\xi = 0$. The hyperbolic ones correspond, after a change of variables at each point, to $\xi_1^2 - (\xi_2^2 + \cdots + \xi_n^2)$, while the parabolic case corresponds, after a change of variables to $\xi_1 + \xi_2^2 + \cdots + \xi_n^2$.

This classification was subsequently extended to linear PDE's of arbitrary order, to nonlinear equations, and to systems. It provides the basic framework in terms of which the theory of PDE's has been systematically studied. Indeed, there are several such theories corresponding to this basic system of classification, including the theory of elliptic equations, hyperbolic equations, parabolic equations and many borderline cases.

Continuing the work of Volterra on the wave equation, Hadamard built up in the 1920's, a systematic theory of the solution of the Cauchy problem for linear second order hyperbolic equations in an arbitrary number of dimensions, including the famous Huygens property for the wave equation in an odd number of space dimensions. In general, solutions of hyperbolic equations depend only on the Cauchy data in a finite domain, the cone of dependence. In the case of the wave equation in odd space dimension the solution depends only on the Cauchy data on the boundary of that cone. The well-known Hadamard conjecture suggests that the wave equation in odd dimensions is the only PDE for which this property holds.

The property of finite dependence for the wave equation is closely connected to the finite speed of propagation in signals governed by equations of this type. The heat equation does not have that property and has infinite speed of propagation. Such considerations are fundamental in the applications of hyperbolic equations in electromagnetic theory where solutions of Maxwell's equation propagate at the speed of light as well as in the equations of relativity where, from the first principles, signals cannot propagate at velocity greater than the speed of light.

The work of Hadamard on second order hyperbolic equations was extended by M. Riesz [RiM] in the late 1940's. Systematic theories of hyperbolic equations and systems of arbitrary order were developed by a number of mathematicians, especially Petrovski [Pet] and Leray [Le4].

11. WEAK SOLUTIONS

Until the 1920's solutions of PDE's were generally understood to be classical solutions, i.e., C^k for a differential operator of order k. The notion of *generalized* or *weak solution* emerged for several different reasons.

The first and simplest occurred in connection with the direct method of the calculus of variations (see Section 6). If one has a variational problem, e.g. the Dirichlet integral E and a minimizing sequence (u_n) for E of smooth functions, it was observed by B. Levi and S. Zaremba that (u_n) is a Cauchy sequence in the Dirichlet norm, and by a simple inequality, in the L^2 norm. Hence, it was natural to introduce the *completion* H under the Dirichlet norm of the space of smooth functions satisfying a given boundary condition. This was a variant of the process began a decade earlier in the case of the L^2 spaces. The space H is a linear subspace of L^2 and is equipped with a different norm. By definition, for any element u of H there is a sequence of smooth functions (u_n) such that grad u_n converges in L^2 to a limit. That limit can be viewed as grad u, interpreted in a generalized sense. This is represented in the work of B. Levi and L. Tonelli and was pursued by many people including K. O. Friedrichs, C. Morrey and others.

The second point of view occurs in problems where the solution is constructed as a limit of an approximation procedure. The estimates on the approximate solutions may not be strong enough to guarantee that the limit is a solution in a classical sense. On the other hand, it may still be possible to show that this limit shares some properties that classical solutions may have, and in particular, relations derived from multiplying the equation by a smooth *testing function* and integrating by parts. This is most familiar in the case of a linear equation; for example a classical solution u of the Laplace equation

$$\Delta u = 0 \qquad \text{in } \Omega \tag{1}$$

satisfies

$$\int \operatorname{grad} u \cdot \operatorname{grad} \varphi = 0, \qquad \forall \varphi \in C_0^\infty(\Omega) = \text{smooth functions with compact support in } \Omega, \tag{2}$$

and

$$\int u \, \Delta\varphi = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{3}$$

The main observation is that (2) makes sense for any function $u \in C^1$ (and even $u \in H$ introduced just above). Relation (3) makes sense if $u \in L^2$ (or even just $u \in L^1_{\mathrm{loc}}$).

In the case of *linear problems*, particularly for elliptic and parabolic equations, it is often possible to show that solutions, even in the weakest sense (3) are classical solutions. The first explicit example is the celebrated Weyl's lemma [We3] proved in 1940 for the Laplace equation. This viewpoint has been actively pursued in the 1960's (see Section 14).

The *existence of weak solutions* is an immediate consequence of the completion procedure described above. The introduction of the concept of weak solutions represents a central *methodological turning point* in the study of PDE's and their BVP's since it presents the possibility of breaking up the investigation of PDE's into 2 steps:

(1) *Existence* of weak solutions.

(2) *Regularity* of weak solutions.

In many cases the second step turns out to be technically difficult or even impossible; sometimes one can obtain only partial regularity. This is especially the case in nonlinear equations. Among the earliest and most celebrated examples are the Navier–Stokes equation:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \nu \, \Delta u_i + \displaystyle\sum_j u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial p}{\partial x_i}, \qquad 1 \leqslant i \leqslant n, \\ \operatorname{div} u = \displaystyle\sum_i \frac{\partial u_i}{\partial x_i} = 0 \end{cases} \tag{4}$$

and the Euler equation:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \displaystyle\sum_j u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial p}{\partial x_i}, \qquad 1 \leqslant i \leqslant n \\ \operatorname{div} u = 0 \end{cases} \tag{5}$$

both describing incompressible fluid flows; the Euler equation is the nonviscous limit of (4).

Local existence and uniqueness (i.e., for a small time interval) of a classical solution for the Euler equation was established beginning with the work of L. Lichtenstein [Li4] in 1925 and more recent contributions by V. Arnold [Arn] (1966), D. Ebin and J. Marsden [E-M] (1970), J. P. Bourguignon and H. Brezis [B-B] (1974) and R. Temam [Te1] (1975). In 2-d ($=2$ space dimensions) the existence of a *global* (i.e., for all time) *classical* solution was treated by W. Wolibner [Wo] in 1933 and completed by T. Kato [Ka2] in 1967. The existence of global classical solutions in 3-d is open.

For the Navier–Stokes equation the existence of a *weak global* solutions (with given initial condition) was obtained first by J. Leray in 1933 (see [Le1,2,3]) and in a slightly different form by E. Hopf [Hop2] in 1950. In 2-d such solutions have been shown to be regular; see [L1]. In 3-d the regularity and the uniqueness of weak solutions is one of the most celebrated open problems in PDE's. For a detailed presentation of the Navier–Stokes equation see e.g. the books of O. Ladyzhenskaya [L1] and R. Temam [Te2].

For some other well-known physical models, such as the theory of nonlinear hyperbolic conservation laws, for example Burger's equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,$$

weak solutions can be defined and are *not regular*, i.e., discontinuities may appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves with important implications in physics (see Section 20).

12. SOBOLEV SPACES

An important systematic machinery to carry through the study of solutions of PDE's was introduced by S. L. Sobolev in the mid 1930's: the definition of new classes of function spaces, the *Sobolev spaces*, and the proof of the most important property, the *Sobolev imbedding theorem* (see [So1,2]).

In a contemporary notation the space $W^{m,p}(\Omega)$ consists of functions u in the Lebesgue space $L^p(\Omega)$, $1\leqslant p<\infty$, having generalized derivatives of all orders, up to m in $L^p(\Omega)$, i.e., there exist functions u_α in $L^p(\Omega)$ such that

$$\int u\,D^\alpha\varphi=(-1)^{|\alpha|}\int u_\alpha\varphi \qquad \forall\varphi\in C_0^\infty(\Omega) \qquad \forall\alpha \text{ with } |\alpha|\leqslant m$$

where α is a multi-index, $\alpha=(\alpha_1, \alpha_2, ..., \alpha_n)$,

$$D^\alpha \varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}},$$

and $|\alpha| = \sum \alpha_i$. When $u \in W^{m,p}(\Omega)$, the functions u_α are called the generalized derivatives $D^\alpha u$ of u.

Another possible approach to such spaces would consist of defining them as the completion of smooth functions with respect to the norm

$$\|u\|^p_{W^{mn,p}} = \sum_{|\alpha| \leqslant m} \|D^\alpha u\|^p_{L^p}.$$

The equivalence of the two definitions for general domains was established in 1964 by N. Meyers and J. Serrin [M-S].

The most important result in the theory of Sobolev spaces concerns inequalities relating the various Sobolev norms. A major precursor is the Poincaré inequality from 1894, [Po2]:

$$\left\| f - \fint f \right\|_{L^2} \leqslant C \, \|\text{grad } f\|_{L^2}$$

(where $\fint f$ denotes the average of f). In a more general form the Sobolev imbedding theorem provides a link between $W^{m,p}$ and $W^{j,r}$ for $j<m$ and $r>p$ (under suitable mild regularity condition on the boundary). The precise form asserts that

$$W^{m,p}(\Omega) \subset W^{j,r}(\Omega)$$

with

$$\|u\|_{W^{j,r}} \leqslant C \, \|u\|_{W^{m,p}}$$

and

$$\frac{1}{r} = \frac{1}{p} - \frac{m-j}{n},$$

provided $r>0$ and Ω is bounded and smooth. Moreover if $s<r$, this imbedding of $W^{m,p}(\Omega)$ into $W^{j,s}(\Omega)$ is *compact*.

If $r<0$ a variant of the above states that

$$W^{m,p}(\Omega) \subset C^{k,\alpha}(\bar{\Omega})$$

and

$$\|u\|_{C^{k,\alpha}} \leqslant C \, \|u\|_{W^{m,p}}$$

where k is the integer part of $(m-(n/p))$ and $\alpha = m-(n/p)-k$.

In this context the concept of generalized derivatives and generalized solutions of PDE's was placed on a firm foundation. Together with the L^p spaces, the Sobolev spaces have turned out to be one of the most powerful tools in analysis created in the 20th century. They are commonly used and studied in a wide variety of fields of mathematics ranging from differential geometry and Fourier analysis to numerical analysis and applied mathematics. For a basic presentation of Sobolev spaces, see e.g. the book of R. A. Adams [Ad]. For more sophisticated results on Sobolev spaces, see the books of V. Mazya [Maz2] and D. R. Adams and L. I. Hedberg [A-H].

13. THE SCHWARTZ THEORY OF DISTRIBUTIONS

Laurent Schwartz, in his celebrated book "La théorie des distributions" (1950) [Scw] presented the generalized solutions of partial differential equations in a new perspective. He created a calculus, based on extending the class of ordinary functions to a new class of objects, the distributions, while preserving many of the basic operations of analysis, including addition, multiplication by C^∞ functions, differentiation, as well as, under certain restrictions, convolution and Fourier transform. The class of distributions (on $\mathbb{R}^n$), $\mathscr{D}'(\mathbb{R}^n)$, includes all functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and any distribution T has well defined derivatives of all orders within that class. In particular, any continuous function (not necessarily differentiable in the usual sense) has a derivative in $\mathscr{D}'$. If

$$L = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, D^\alpha$$

is a linear differential operator with smooth coefficients, then $L(T)$ is well defined for any distribution T and $L(T)$ is again a distribution.

The definition of distributions by L. Schwartz is based on the notion of duality of topological vector spaces. The space $\mathscr{D}'(\mathbb{R}^n)$ consists of continuous linear functionals on $C_0^\infty(\mathbb{R}^n)$, i.e., the *dual space* of the space of testing functions $C_0^\infty(\mathbb{R}^n)$ equipped with a suitable topology involving the convergence of derivatives of all orders. This definition implies that each distribution T can be represented locally as a (finite) sum of derivatives (in the distribution sense) of continuous functions, i.e.,

$$T(\varphi) = \sum_{|\alpha| \leqslant m} \int f_\alpha\, D^\alpha \varphi \qquad \forall \varphi \in C_0^\infty$$

for some continuous functions f_α and some m.

This theory systematized and made more transparent related earlier definitions of generalized functions developed by Heaviside, by Hadamard, Leray and Sobolev in PDE, and by Wiener, Bochner and Carleman in Fourier analysis.

Other significant motivations for the theory of distributions included:

(a) Giving a more transparent meaning to the notion of elementary (or fundamental) solution E of an elliptic operator L, which in the language of the theory of distributions is

$$L(E)=\delta_0$$

where δ_0 is the Dirac measure at 0, i.e., $\delta_0(\varphi)=\varphi(0)$.

(b) D'Alembert's solution of the 1-d wave equation is $u(x, t)=f(x+t)$ $+g(x-t)$. This u is a classical solution if f, g are smooth and u is a distribution solution if f, g are merely continuous (or just L^1_{loc}).

In terms of the theory of distributions, Sobolev spaces can be defined as

$$W^{m,p}=\{u\in L^p;\ D^\alpha u\in L^p \text{ in the sense of distributions}, \forall\alpha,\ |\alpha|\leqslant m\}.$$

Many of the applications of the theory of distributions have been in problems formulated in terms of Sobolev spaces. However there are other significant classes which play an important role. An example is the space of *functions of bounded variation*

$$BV=\left\{u\in L^1;\ \frac{\partial u}{\partial x_i} \text{ is a measure}, \forall i=1, 2, ..., n\right\}.$$

This definition clarified a complex field of competing notions (in particular in the works of L. Tonelli and L. Cesari). The BV space is very useful in the calculus of variations (e.g. geometric measure theory, fracture mechanics and image processing) as well as in the study of shock waves for nonlinear hyperbolic conservation laws (see Section 20).

For a special subclass of distributions, the tempered distributions, $\mathscr{S}'$, L. Schwartz defined a Fourier transform which carries $\mathscr{S}'$ into $\mathscr{S}'$. The class $\mathscr{S}'$ is defined again as the dual space of a larger class of test functions

$$\mathscr{S}(\mathbb{R}^n)=\{u\in C^\infty(\mathbb{R}^n);\ |x|^m\, D^\alpha u(x)\in L^\infty(\mathbb{R}^n), \forall m, \forall\alpha\}.$$

Using the class $\mathscr{S}'$ one can exploit the very important fact that the Fourier transform of $D^\alpha u$ is

$$\mathscr{F}(D^\alpha u)(\xi)=(i)^{|\alpha|}\,\xi^\alpha\mathscr{F}(u)(\xi)$$

where

$$\mathscr{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

For a linear differential operator L with *constant* coefficients

$$L = \sum a_\alpha D^\alpha$$

the study of the solution of the equation $Lu = f$, after Fourier transform, reduces to the study of an algebraic equation

$$P(\xi)(\mathscr{F}u) = \mathscr{F}f$$

where $P(\xi) = \sum a_\alpha (i)^{|\alpha|} \xi^\alpha$. Thus, this problem is equivalent to the study of division by polynomials in various spaces of distributions. This viewpoint and, in addition, the introduction of the Fourier Transform in the complex domain (as first suggested by Leray [Le4]), has been the subject of intensive investigation beginning in the mid-1950's in the work of L. Ehrenpreis [Eh], B. Malgrange [Mal] and L. Hörmander [Hor1].

This gives rise to a theory of *local solvability* for linear PDE's with *constant coefficients*, which has since been generalized to a theory of local solvability for equations with *variable coefficients* (see H. Lewy [Lew], A. Calderon [Cal2], L. Nirenberg and F. Treves [N-T], R. Beals and C. Fefferman [B-F]).

In the ensuing decades the theory of distributions provided a unifying language for the general treatment of solutions of PDE's. In addition to its universal use in analysis, it has been widely adopted in many areas of engineering and physics. An important extension of the machinery of the theory of distributions was the development of the theory of analytic functionals by Sato and his school and other related theories of hyperfunctions. For a general treatment of distribution theory in the theory of PDE, see [Hor4]. For some other topics on the use of distribution theory in PDE's, see [G-S].

14. HILBERT SPACE METHODS

One of the great mathematical advances in the 1930's was the development in a conceptually transparent form of the theory of self-adjoint linear operators and the more general framework for linear functional analysis in the work of S. Banach and his school. Though the first was based on earlier work of Fredholm, Hilbert and F. Riesz on integral equations, the reformulation of the basic principle of quantum mechanics in operator theoretic terms gave an enormous impetus to the more sophisticated development of operator theory in Hilbert spaces, in geometric and analytic forms. At the same time, except for isolated work of K. O. Friedrichs and H. Weyl, few applications were made of these ideas to PDE's. This situation changed very quickly in the late 1940's especially because of the early work of M. I. Visik (1951) under the influence of I. M. Gelfand. M. I. Visik [Vi1] considered the formulation of the Dirichlet problem for a general nonselfadjoint uniformly elliptic linear operator (not necessarily second order). When written in generalized divergence form, such operator becomes

$$Lu = \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} D^\alpha (a_{\alpha\beta}(x)\, D^\beta u) \tag{6}$$

where

$$\operatorname{Re} \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x)\, \xi^\alpha \xi^\beta \geqslant c_0\, |\xi|^{2m} \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad c_0 > 0. \tag{7}$$

These results were sharpened in the work of L. Gårding (1953) [Gå1] as well as in related works of F. Browder [Bro1], K. O. Friedrichs [Fd], P. Lax and A. Milgram [L-M], and J. L. Lions [Lio1]. Gårding's most important contribution was to introduce the explicit use of Fourier analysis into this field and, in particular, the central role of Plancherel's theorem (1910) which states that the Fourier transform is a unitary mapping of $L^2(\mathbb{R}^n)$ into itself. As we have already noted the Fourier transform $\mathscr{F}$ carries the differential operator D^α into the operator of multiplication by $(i)^{|\alpha|}\, \xi^\alpha$. In terms of this operation the Sobolev space $H^m = W^{m,2}$ becomes, under Fourier transform,

$$W^{m,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n);\ \xi^\alpha \mathscr{F}(u) \in L^2(\mathbb{R}^n),\ \forall \alpha,\ |\alpha| \leqslant m\},$$

with equivalence of norms, namely,

$$\|u\|^2_{m,2} = \|u\|^2_{W^{m,2}} \simeq \sum_{|\alpha| \leqslant m} \|\xi^\alpha \mathscr{F}(u)\|^2_{L^2},$$

giving an alternative perspective on the Sobolev imbedding theorem for $p = 2$. As opposed to the Sobolev space $W^{m,p}$, $p \neq 2$, $W^{m,2}$ is a Hilbert space with inner product

$$(u, v) = \sum_{|\alpha| \leqslant m} \int D^\alpha u\, \overline{D^\alpha v}.$$

In terms of this formalism, L. Gårding established the well-known *Gårding inequality*: If Ω is a bounded domain in $\mathbb{R}^n$ and if $W_0^{m,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,2}(\Omega)$, then for every L of the form (6) with the top order coefficients $a_{\alpha\beta}$ satisfying (7), uniformly continuous on Ω, and all coefficients bounded, then there exist constants $c_0 > 0$ and k_0 such that

$$\operatorname{Re} \int (Lu)\,\bar{u} \geqslant c_0\, \|u\|_{m,2}^2 - k_0\, \|u\|_{0,2}^2 \qquad \forall u \in W_0^{m,2}(\Omega).$$

This inequality plays an essential role in reducing the existence problem to standard results in Hilbert space theory.

The classical Dirichlet problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ D^\alpha u &= 0 && \text{on } \partial\Omega, \quad \forall \alpha,\ |\alpha| < m, \end{aligned}$$

can be extended from smooth solutions $u \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$ to generalized solutions $u \in W_0^{m,2}(\Omega)$ satisfying

$$\sum_{\alpha,\beta} \int a_{\alpha\beta} D^\alpha u\, \overline{D^\alpha v} = \int f \bar{v} \qquad \forall v \in C_0^\infty(\Omega)$$

and a fortiori for all $v \in W_0^{m,2}(\Omega)$. This latter problem is called the generalized Dirichlet problem and follows the same pattern as the completion process described in Section 11 for the Dirichlet problem associated with the Laplace equation.

By the Frechet–Riesz representation theorem there exists a bounded linear operator A from $H = W_0^{m,2}(\Omega)$ to H such that

$$(Au, v)_H = (Lu, v)_{L^2} \qquad \forall u, v \in H.$$

Similarly there exists an element $g \in H$ such that

$$(f, v)_{L^2} = (g, v)_H \qquad \forall v \in H$$

and a compact linear map C of H into H such that

$$(u, v)_{L^2} = (Cu, v)_H \qquad \forall u, v \in H.$$

The generalized Dirichlet problem is immediately translatable into the functional equation

$$Au = g, \qquad u \in H$$

and the Gårding inequality into

$$\operatorname{Re}(Au, u)_H \geqslant c_0 \|u\|_H^2 - (Cu, u) \qquad \forall u \in H.$$

If $C = 0$, we may apply the Lax–Milgram lemma [L-M], which asserts that every bounded linear operator A from H to H for which

$$\operatorname{Re}(Au, u) \geqslant c_0 \|u\|_H^2 \qquad \forall u \in H,$$

is an isomorphism of H onto itself. In this case the Dirichlet problem is solvable uniquely for every f. In the general case

$$A = A_0 + C$$

where A_n is an isomorphism and C is compact. By the classical theory of F. Riesz [RiF], A is a Fredholm operator of index zero. In particular, one has the Fredholm alternative, namely the equation $Au = f$ has a solution if and only if f is orthogonal to the finite dimensional nullspace of A^*, $N(A^*)$, and $\dim N(A^*) = \dim N(A)$.

To obtain the completeness of the eigenfunctions of the Dirichlet problem for a formally self-adjoint A of order $2m$ one may apply the spectral decompositions of compact self-adjoint operators in Hilbert spaces. One introduces a new inner product on H given by

$$[u, v] = (Au, v)_H + k(u, v)_{L^2}.$$

By Gårding's inequality this is a scalar product if k is sufficiently large and the associated norm is equivalent to the original norm on H. If one introduces the operator C by

$$[Cu, v] = (u, v)_{L^2},$$

C is a compact self-adjoint operator in H with respect to the new inner product. The eigenvalue problem $Lu = \lambda u$, $u \in W_0^{m,2}(\Omega)$, is equivalent to the functional equation

$$u = (k + \lambda)\, Cu, \qquad u \in H$$

and therefore the spectral structure of C goes over to the eigenvalue decomposition for L. The asymptotic distribution of eigenvalues for the Dirichlet problem has been extensively studied following the initial result of H. Weyl (1912) [We1] (see Section 18).

Another equivalent viewpoint of treating the Dirichlet problem lies in using the *duality structure* of Banach spaces more explicitly. Following a definition introduced by J. Leray [Le4] in the treatment of hyperbolic equations and independently by P. Lax [La1] in the treatment of elliptic equations, one can define the Sobolev space $W^{-m,2}(\Omega)$ as the conjugate space of $W_0^{m,2}(\Omega)$, where this new space is considered as a space of distributions. Similarly one defines $W^{-m,p'}(\Omega)$ as the conjugate space of $W_0^{m,p}(\Omega)$ where $p' = p/(p-1)$. In the case $p=2$ the Riesz representation theorem establishes an isomorphism between $W_0^{m,2}(\Omega)$ and $W^{-m,2}(\Omega)$. It is this isomorphism which we apply above to represent the mapping L, which is more naively defined as a mapping of $W_0^{m,2}(\Omega)$ onto $W^{-m,2}(\Omega)$ by the new operator A mapping $W_0^{m,2}(\Omega)$ into itself. For the extensions of this procedure to a nonlinear setting, where in general $p \neq 2$, see Section 21.

These results on the existence (and uniqueness) of solutions of the generalized Dirichlet problem must be supplemented—when all data are smooth—by results on the regularity of these generalized solutions to obtain a classical solution. Such results involve both regularity in the interior as well as regularity up to the boundary. Results of the first kind were obtained by:

1. Use of fundamental solutions for elliptic operators of higher order as established by F. John [J1], generalizing classical results of E. E. Levi [Le] in the analytic case.

2. Use of Friedrichs' method of mollifiers involving convolutions of the given u with a sequence of smoothing kernels; see [Fd].

3. Use of the Lichtenstein finite difference method as revived by Morrey [Mor2].

The first two methods apply to a somewhat broader problem, namely proving that all distribution solutions of $Lu = f$, i.e., $u \in \mathscr{D}'(\Omega)$ satisfies $Lu = f$ in the distribution sense, are C^∞ when L is elliptic with smooth coefficients and f is C^∞. When L is the Laplacian and $u \in L^2$ this result was established by H. Weyl [We3] in 1940, and this so-called *Weyl lemma* was the inspiration for the whole field of studying the regularity of distribution solutions of elliptic equations. This is the central example of a situation where every distribution solution u of the equation $Lu = f$ with $f \in C^\infty$ must lie in C^∞. Such a property has been extensively studied for general operators under the name of *hypoellipticity*.

These results were also applied to obtain solutions of equations of evolution involving L of the parabolic and generalized wave equation type; see Section 17.

A related development of major importance was the application of *energy methods* to the study of the Cauchy problem for linear strictly hyperbolic PDE's and systems of PDE's. After initial work in 1938 by J. Schauder on second order hyperbolic equations and later work by K. O. Friedrichs on symmetric hyperbolic systems, the full generality of the pre-war results of Petrovski [Pet] was recovered and amplified by J. Leray [Le4] using global energy estimates. These estimates were later localized by L. Gårding [Gå3].

15. SINGULAR INTEGRALS IN L^p: THE CALDERON–ZYGMUND THEORY

An essential tool in the study of regularity properties of solutions of PDE's has been the L^p theory of singular integral operators developed by Calderon and Zygmund in 1952. Singular integral operators on $\mathbb{R}^n$ are operators of the form

$$(Sf)(x) = pv \int_{\mathbb{R}^n} \frac{K(x-y)}{|x-y|^n} f(y)\, dy = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{K(x-y)}{|x-y|^n} f(y)\, dy,$$

where $K(x) = k(x/|x|)$ and k satisfies some smoothness condition together with

$$\int_{S^{n-1}} k(\xi)\, d\sigma(\xi) = 0.$$

Two principal examples motivate this theory:

1. The *Hilbert transform* $\mathscr{H}$ which is an important tool in Fourier analysis on $\mathbb{R}$ corresponds to $n=1$, $k(+1) = +1$ and $k(-1) = -1$.

2. If E is the fundamental solution for the Laplace operator in $\mathbb{R}^n$, i.e.,

$$E(x) = \begin{cases} c/|x|^{n-2} & \text{if} \quad n>2, \\ c \log(1/|x|) & \text{if} \quad n=2, \end{cases}$$

then for every i, j

$$K(x) = |x|^n \frac{\partial^2 E}{\partial x_i\, \partial x_j}$$

satisfies the above conditions. In view of the results of Section 14, for any solution u of the Laplace equation $-\Delta u = f$, $u - (E * f)$ is harmonic and thus C^∞. Therefore the regularity properties of u are the same as those of $(E * f)$. Moreover

$$\frac{\partial^2}{\partial x_i\, \partial x_j}(E * f) = \frac{\partial^2 E}{\partial x_i\, \partial x_j} * f,$$

at least formally; more precisely $\partial^2 E/\partial x_i\, \partial x_j$ is not an L^1 function and thus the convolution cannot be defined as the integral of an L^1 function. It must be considered as a principal value (this is already true in the case of the Hilbert transform $\mathscr{H}$). Singular integral operators have been considered in connection with PDE's in the works of F. E. Tricomi (1926-28), G. Giraud (1934) and especially S. G. Mikhlin starting in 1936; see [Mik].

For $\mathscr{H}$, M. Riesz, in 1927, proved that $\mathscr{H}$ carries $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for all $1<p<\infty$. From the point of view of PDE's, the most important contribution of Calderon–Zygmund, in their celebrated 1952 paper [C-Z], was to generalize this result about $\mathscr{H}$ to show that every singular integral operator S as above maps $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and satisfies the inequality

$$\|Sf\|_{L^p} \leqslant C_p \|f\|_{L^p} \qquad \forall f \in L^p(\mathbb{R}^n).$$

Applications of this result were made to important problems in PDE's, within a few years, by Calderon and others. In particular, Calderon [Cal1] obtained the uniqueness of the Cauchy problem for operators with simple characteristics. The Calderon–Zygmund estimates were applied by L. Bers to obtain basic theorems about Teichmuller spaces. In addition, Calderon developed representation theorems for solutions of BVP, for general linear elliptic equations in terms of singular integrals applied to the boundary data.

The calculus was extended to singular integral operators

$$(Sf)(x) = pv \int \frac{K(x, x-y)}{|x-y|^n}\, dy$$

where, for each x, $K(x, \cdot)$ is a singular integral kernel in the above sense. The symbol σ of these operators plays a strategic role, where

$$\sigma(x, \xi) = \mathscr{F}(K(x, \cdot))(\xi).$$

The composition of two such operators is a singular integral operator modulo smoothing operators and its symbol is the product of the two symbols. When in the early 1960's, Atiyah and Singer [A-S] attacked the problem formulated by Gelfand of calculating the index of a linear elliptic system of differential operators acting on a vector bundle over a compact manifold in terms of topological invariants, the technical framework of their theory, in terms of analysis, was the deformation of systems of differential operators through systems of operators whose coefficients were singular integral operators. In this study the principal tool was the use of the symbol of the singular integral operators and the fact that the composite operators define Fredholm mappings in appropriate function spaces, which vary continuously with the symbol. It was this application of the singular integral operators which gave rise in 1965 to the definition of pseudo-differential operators by J. J. Kohn and L. Nirenberg [K-N], (and also by R. T. Seeley, L. Hörmander, A. Unterberger and J. Bokobza) thereby providing a unified framework for the concepts of singular integral operators and differential operators with powerful rules of computation. This calculus also includes the one devised in 1927 by H. Weyl [We2] in connection with problems of quantum mechanics.

More explicitly, *the pseudo-differential operator* associated with the symbol $\sigma(x, \xi)$ is given by

$$(Pf)(x) = \int e^{ix \cdot \xi} \sigma(x, \xi)(\mathscr{F}f)(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \iint e^{i(x-y) \cdot \xi} \sigma(x, \xi)\, f(y)\, dy\, d\xi.$$

Note that differential operators correspond to symbols σ which are polynomials in the ξ variable while the singular integral operators described above correspond to symbols σ which are homogeneous of order zero in ξ.

A more general class of transformations, called *Fourier integral operators*, is given with respect to a phase function $\varphi(x, y, \xi)$ by

$$\frac{1}{(2\pi)^{n/2}} \iint e^{i\varphi(x, y, \xi)} a(x, y, \xi)\, f(y)\, dy\, d\xi.$$

The theory of such transformations, which has been initiated by P. Lax [La2] and V. P. Maslov [Mas], and developed by L. Hörmander [Hor3], Yu. V. Egorov, J. J. Duistermaat, R. Melrose and others, provides a powerful tool for studying solutions of linear hyperbolic equations. An important use of both transformations is the study of propagation of singularities along their bicharacteristics in conjunction with the important notion of wave front set, first introduced by Sato for hyperfunctions and then by L. Hörmander for distributions. This area of attack on solutions of PDE's is usually called microlocal analysis.

Another important tool, the theory of paradifferential operators, was introduced by J. M. Bony [Bon] for the study of propagation of singularities for solutions of nonlinear hyperbolic equations.

A significant strengthening of the Calderon–Zygmund theory was the development of the theory of commutators with Lipschitz continuous kernels initiated by Calderon and continued by R. Coifman, Y. Meyer and A. MacIntosh; see e.g. [Me, Vol. 3]. An interesting domain of application to PDE's is the work of C. Kenig [Ken] on elliptic equations in irregular domains.

For a detailed account, see the books of L. Hörmander [Hor4], J. J. Duistermaat [Du], F. Treves [Tre], M. Taylor [Ta1] [Ta2], Yu. V. Egorov and M. A. Shubin [E-S] and E. Stein [Ste].

16. ESTIMATES FOR GENERAL LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

In the tradition of J. Schauder and his predecessors (see Section 8) a general treatment of solvability and a priori estimates for higher order linear elliptic problems was carried out in the late 1950's. The class of problems for which such results hold was described by the Soviet mathematicians Ya. B. Lopatinski [Lo] and Z. Shapiro [Sh]. In terms of the characteristic polynomial of the elliptic operator

$$L = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, D^\alpha$$

and the system of boundary operators

$$B_j = \sum_{|\beta| \leqslant m_j} b_\beta(x)\, D^\beta, \qquad m_j \leqslant m, \quad j = 1, 2, ..., m,$$

an algebraic condition, at all boundary points, involving the characteristic polynomials

$$a(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x)\, \xi^\alpha$$

and

$$b_j(x, \xi) = \sum_{|\beta| = m_j} b_\beta(x)\, \xi^\beta$$

is essentially equivalent to the solvability (in a reasonable sense) of the problem

$$\begin{cases} L_u = f & \text{in } \Omega \\ B_j u = g_j & \text{on } \partial\Omega, \qquad j = 1, 2, ..., m. \end{cases}$$

A particular case is the Dirichlet BVP for a uniformly elliptic operator of order $2m$; here $B_j = \partial^j/\partial n^j$, $j = 0, 1, ..., m-1$.

The study of such equations (and systems) in various function spaces, such as C^α, L^p, etc..., was begun by a number of mathematicians, culminating in the celebrated and very general paper by S. Agmon, A. Douglis and L. Nirenberg [A-D-N]. Following the example of L. Lichtenstein, Kellogg and Schauder in the case of the Dirichlet problem for second order equations, the technical study of the theory is reduced to a model problem: the representations of solutions of the constant coefficient operators in a half-space with homogenous, constant coefficient, boundary conditions. Such representations were given in the most explicit form in the so-called Poisson kernel. Estimates for such problems can be perturbed to yield local estimates for variable coefficient problems under suitable hypotheses on the coefficients (C^α for C^α estimates and uniformly continuous for L^p estimates). The estimates are of the following type

$$\|u\|_{C^{2m,\alpha}(\bar{\Omega})} \leqslant C\left(\|f\|_{C^{0,\alpha}(\bar{\Omega})} + \|u\|_{C^{0,\alpha}(\bar{\Omega})} + \sum_j \|g_j\|_{C^{2m-mj,\alpha}(\partial\Omega)} \right)$$

and

$$\|u\|_{W^{2m,p}(\Omega)} \leqslant C\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \sum_j \|g_j\|_{W^{2m-mj-(1/p),\, p}(\partial\Omega)} \right).$$

Here, the boundary term involves a *fractional Sobolev* norm. When $0<s<1$ the norm in $W^{s,p}$ over a domain or manifold of dimension d is given by

$$\|u\|^p_{W^{s,p}} = \iint \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\, dx\, dy + \int |u(x)|^p\, dx.$$

Similar estimates can be obtained for higher order derivatives and the original estimates can be used to derive existence theorems in various function spaces (as well as the fact that such operators are Fredholm); see F. Browder [Bro3,4], M. Schechter [Sce], the book of J. L. Lions and E. Magenes [Li-Ma] and the references therein.

An additional result involves the application of *interpolation* procedures in Sobolev spaces. The most systematic form is the Gagliardo–Nirenberg inequality [Ga1,2] and [Ni2]. An especially useful case states that

$$\|u\|^2_{W^{1,q}} \leqslant C\, \|u\|_{W^{2,p}}\, \|u\|_{L^r}$$

where

$$\frac{1}{q} = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{r}\right).$$

The combination of the a priori estimates with interpolation properties of Sobolev spaces has been an important device in studying nonlinear problems and has made the calculus in Sobolev spaces an essential tool.

17. LINEAR EQUATIONS OF EVOLUTION: THE HILLE–YOSIDA THEORY

The classical BVP's of mathematical physics include, besides the elliptic equations, the initial BVP for the heat equation and the Cauchy problem for the wave equation; in addition, following the development of quantum mechanics, the initial value problem for the Schrödinger equation.

All these problems can be written in a common form:

$$\begin{cases} \dfrac{du}{dt} = Au, & t \in [0, \infty) \\ u(0) = u_0 \end{cases} \tag{8}$$

where:

(1) for the heat equation $A = \Delta$,

(2) for the wave equation, $u = \binom{u_1}{u_2}$ is a vector, $u_1 = u$, $u_2 = \partial u/\partial t$ and A is the matrix

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

(3) for the Schödinger equation $A = i(\Delta + V)$, where $V(x)$ is a potential function.

One can replace the Laplace operator Δ, in the above examples, by a general elliptic operator provided one establishes appropriate results on the spectral properties of L under the given homogenous boundary condition. For problems (2) and (3) this traditionally means that L is formally self-adjoint, so that the corresponding operators in Hilbert spaces are Hermitian and have real spectrum.

A general treatment of initial value problems of this type was given in 1948, independently by E. Hille [H-P] and K. Yosida [Yo1,2]. Their theorem (in a slightly generalized form) asserts that if X is a Banach space and $A\colon D(A) \subset X \to X$ is a possibly unbounded closed linear operator such that

$$\begin{cases} (A - \lambda I)^{-1} \quad \text{exists for all } \lambda > \omega \text{ and satisfies} \\ \|(A - \lambda I)^{-n}\| \leqslant M(\lambda - \omega)^{-n} \qquad \text{for all} \quad \lambda > \omega, \end{cases} \tag{9}$$

for some constants ω and M, then (8) has a unique solution $u(t)$ for each $u_0 \in D(A)$. The mapping $U(t)\colon u_0 \mapsto u(t)$ satisfies

$$\|U(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0 \tag{10}$$

as well as the *semi-group* property

$$U(0) = I, \qquad U(t+s) = U(t)\, U(s), \qquad \forall t, s \geqslant 0.$$

Moreover, every continuous semi-group satisfying (10) is obtained in this way, for some operator A, called the infinitesimal generator of $U(t)$. If both A and $-A$ satisfy (9), e.g. $A = iH$ where A is a Hermitian operator in a Hilbert space, then equation (8) can be solved both for positive and negative time and $U(t)$ is a one-parameter group. Physically this corresponds to *time reversibility*; it occurs, for example, in the wave and Schrödinger equations, but *not* in the heat equation.

In applying Hille–Yosida theory to the concrete examples mentioned above, one obtains results on $(A - \lambda I)^{-1}$ by showing that the equation

$$Au - \lambda u = f$$

has a unique solution u in $D(A)$ for any given $f \in X$. This is an existence (uniqueness) statement for an elliptic stationary problem and is treated by the methods of Sections 8, 14, 16. The interested reader will find a detailed presentation of the theory of semigroups and its applications in the books of E. B. Davies [Da1], J. Goldstein [Go], A. Pazy [Pa], M. Reed and B. Simon [R-S], Vol 2.

18. SPECTRAL THEORIES

The considerations above provide one of the principal motivations for the study of the spectral theory of elliptic operators under homogenous boundary conditions, which has been extensively developed over the 20th century in a number of different directions.

For some classical operators, particularly the Schrödinger operator $A = -\Delta + V$, this investigation began in the work of Friedrichs and Rellich (in the 1930's and 40's) and was actively pursued by T. Kato (in the 1950's and 60's) and many others. The main purpose is to study the effect on the spectrum of small perturbations of A (e.g. on the potential V). The spectral properties of the operator A are closely related to the asymptotic properties of $U(t)$ as $t \to \infty$, which have been studied under the name of *scattering theory*. For the time dependent Schrödinger equation, this is the classical scattering problem of quantum mechanics. We refer to the books of T. Kato [Ka1], L. Hörmander [Hor4], M. Reed and B. Simon [R-S]. A related problem has been extensively investigated by P. Lax and R. Phillips [L-P] for the wave equation in exterior domains; further results were obtained by C. Morawetz and W. Strauss [Mo-St] as well as by J. Ralston, R. Melrose and J. Rauch.

Among the many developments in the spectral theory of elliptic self-adjoint operators (as well as more general linear PDE's) let us mention the theory of singular eigenfunctions expansion (analogous to the Fourier integrals) for operators without compact resolvents. If A is such an operator, by the abstract spectral theorem in Hilbert space

$$A = \int \lambda \, dE_\lambda$$

where $\{E_\lambda\}$ is the spectral measure corresponding to A. The problem of singular eigenfunction expansions is that of expressing E_λ as a transform using eigenfunctions of A. This was initiated in a paper of Mautner [Mau] (1952) and developed in full by F. Browder [Bro2], L. Gårding [Gå3] and I. M. Gelfand (see the book of Gelfand and Shilov [G-S, Vol. 3]).

In the case of a compact resolvent an important topic of investigation is the *asymptotic distribution of eigenvalues* begun by H. Weyl for the Laplacian in his famous paper [We1] in 1912. The question was posed by the physicist H. Lorenz (in 1908) as an important tool in proving the equipartition of energy in statistical mechanics. H. Weyl established the necessary result, i.e., if $N(\lambda)$ denotes the number of eigenvalues $\leqslant \lambda$, then

$$N(\lambda) \simeq c_n \lambda^{n/2} \operatorname{vol}(\Omega) \qquad \text{as} \quad \lambda \to \infty,$$

where c_n depends only on n.

Weyl's method applied the minimax principle for eigenvalues of Hermitian matrices introduced by Fisher [Fis] and extended by Weyl to integral operators. This method used a decomposition of the domain into pieces on which the eigenvalue problem can be solved explicitly. (A similar approach based on the minimax principle was used later by Courant to obtain the first estimates on the order of magnitude of the error term, (see [C-H], Vol. I).

An important transformation of the problem was carried through by Carleman [Car] in 1934 who began the estimation of the *spectral function*

$$e(x, y, \lambda) = \sum_{\lambda_i \leqslant \lambda} e_i(x) \, \overline{e_i(y)}$$

where $\{e_i\}$ is the family of orthonormalized eigenfunctions. The function $e(x, y, \lambda)$ is the kernel of the spectral projection operator E_λ and $N(\lambda) = \int_\Omega e(x, x, \lambda) \, dx$.

Carleman observed that for the Green's function $G(x, y, \lambda)$ of $A + \lambda I$,

$$G(x, y, \lambda) = \int e(x, y, \mu) \frac{d\mu}{\mu + \lambda},$$

and obtained asymptotic estimates on $e(x, y, \lambda)$ by applying Tauberian theorems to corresponding asymptotic estimates for $G(x, y, \lambda)$. Later

Minakshisundaram and Pleijel [M-P] observed that if one uses the fact that the solution of the initial value problem equation $\partial u/\partial t = -Au$ is given by

$$u(t) = U(t)\, u_0, \qquad U(t) = \int e^{-\lambda t}\, dE_\lambda,$$

then a similar Tauberian argument gives asymptotic estimates for $e(x, y, \lambda)$ in terms of estimates for the kernel of $U(t)$. Still later, Hörmander applied an analogous argument for the generalized wave equation

$$\frac{\partial^2 u}{\partial t^2} - Au = 0$$

for which the solution of the Cauchy problem can be expressed in terms of the kernel of the operator

$$\int e^{it\sqrt{\lambda}}\, dE_\lambda.$$

The asymptotics of the spectral function as well as of the *trace* of heat kernel, $\sum e^{-\lambda_i t}$, especially popular among geometers, have attracted much attention, for elliptic operators, even of higher order, and on manifolds. We mention, in particular, the works of B. M. Levitan (1952–55). L. Gårding [Gå2], S. Mizohata and R. Arima (1964), H. P. McKean and I. Singer [M-S] (1967), L. Hörmander [Hor2] (who introduced, in 1968, Fourier integral operators as a tool for estimating remainder terms in the expansion of the spectral function), J. J. Duistermaat and V. W. Guillemin [D-G] (1975), A. Weinstein (1977), R. T. Seeley (1978–1980), Y. Colin de Verdière (1979), V. Ivrii [I] (1980) and others (see a detailed presentation in the books [Hor4] and [Ta1]). For more recent results in spectral theory, see E. B. Davies [Da2] and Safarov–Vassiliev [S-V].

The celebrated problem of M. Kac [Ka] "Can one hear the shape of a drum?", i.e., does the spectrum of the Laplacian fully determine the geometry of the domain? has received a negative answer in 1991 (see [G-W-W]), but the question remains how much of the geometry is recoverable from the spectrum.

A related set of questions, going under the name of *inverse problems*, asks for the determination of the potential $V(x)$ in the Schrödinger operator $(-\Delta + V)$ in terms of the spectral data. This problem was first posed in connection with quantum mechanics and is also of significance in seismology. The positive solution to this problem was achieved in 1-d by the Gelfand–Levitan theory [G-L] in 1951 and eventually, proved to be an essential tool in the analysis of soliton solutions for the $K\,dV$ equation (see Section 20).

Another inverse problem introduced by A. P. Calderon in 1980 asks whether the coefficient function $a(x)$ in the operator $L = \operatorname{div}(a(x) \operatorname{grad})$ can be determined from the knowledge of the mapping which associates to every function φ on $\partial\Omega$ the value of $a(\partial u/\partial n)$ on $\partial\Omega$ where u is the solution of $Lu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$. This problem is of great importance in engineering because, in practice, measurement can only be made on the boundary. Recent results of R. Kohn and M. Vogelius (1984), J. Sylvester and G. Uhlmann (1987) indicate that the answer is positive in dimension $\geqslant 3$ (see [Sy-Uh] and the references therein).

19. MAXIMUM PRINCIPLE AND APPLICATIONS: THE DEGIORGI–NASH ESTIMATES

A characterizing principle for a harmonic function in a domain Ω of $\mathbb{R}^n$ is that, at each x,

$$u(x) = ⨍_{B_r(x)} u(y)\, dy$$

for any ball $B_r(x)$ in Ω, where $⨍$ denotes the average. A consequence is that u cannot assume a maximum value at an interior point unless it is constant. Starting with the work of Paraf in 1892 and continued by Picard and Lichtenstein, this conclusion was extended to second order linear uniformly elliptic operators

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_0\, \partial x_j} + \sum_i a_i(x) \frac{\partial}{\partial x_i} + a_0(x),$$

with smooth coefficients provided that $a_0 < 0$. An important sharpening of this theorem was established by E. Hopf [Hop1] in 1927 without any assumptions of continuity on the coefficients (just boundedness). His result asserts that if $u \in C^2$ satisfies

$$\begin{cases} Lu = f(x) & \text{in } \Omega \\ u = \varphi(x) & \text{on } \partial\Omega \end{cases} \tag{11}$$

with $f \geqslant 0$ in Ω and if u attains a nonnegative maximum M at an interior point in Ω then $u \equiv M$. In particular if u satisfies (11) with $f \geqslant 0$ in Ω and $\varphi \leqslant 0$ on $\partial\Omega$ then $u \leqslant 0$ everywhere in Ω. Thus the map $(f, \varphi) \mapsto u$ is order preserving, i.e., $f_1 \leqslant f_2$ and $\varphi_1 \geqslant \varphi_2$ imply $u_1 \geqslant u_2$. A consequence of this is uniqueness of the solution of (11). The weak assumptions in Hopf's result imply that this result goes over to very general solutions of nonlinear equations.

Starting with S. Bernstein (see Section 7) the maximum principle has provided a decisive instrument in proving a priori estimates and existence. The procedure has always consisted of ingenious choices of auxiliary functions satisfying elliptic partial differential inequalities.

Important early application of the maximum principle was the use of subharmonic functions, that is,

$$u(x) \leqslant \fint_{B_r(x)} u(y)\, dy$$

for all $B_r(x) \subset \Omega$ (or equivalently $\Delta u \geqslant 0$ in the sense of distributions), as a useful concept in potential theory. For example, the solution of $\Delta u = 0$ in Ω with $u = \varphi$ on $\partial\Omega$ coincides with $\sup_{i \in I} u_i$ where $(u_i)_{i \in I}$ denotes the family of all subharmonic functions on Ω such that $u_i \leqslant \varphi$ on $\partial\Omega$. This is called after O. Perron [Per] who initiated this approach in 1923. N. Wiener [Wi] extended this result in 1924 to obtain a necessary and sufficient criterion for proving that, at a given $x_0 \in \partial\Omega$, the above u satisfies $u = \varphi$.

From such considerations one derives a constructive method for solving a class of nonlinear elliptic equations via a monotone iteration, in the presence of an ordered pair of sub and supersolutions.

A related, but sharper result is Harnack's inequality (1887) which states that if u is harmonic in Ω, $u \geqslant 0$ in Ω then for each compact subdomain K,

$$\sup_K u \leqslant C_K \inf_K u$$

where C_K depends only on K. This principle provides a useful compactness property for harmonic functions.

Important progress in this direction was made by E. DeGiorgi [Dg1] in 1957 and subsequently refined by J. Moser [Mos1] and G. Stampacchia [Sta]. The main point is that the maximum principle, as well as Harnack's inequality, hold for second order elliptic operators in divergence form

$$Lu = \sum_{i,j} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_i a_i \frac{\partial u}{\partial x_i} + a_0 u$$

with $a_0 \leqslant 0$, under the very weak assumption that the coefficients a_{ij} are bounded measurable and satisfy a uniform ellipticity condition

$$\sum a_{ij}(x)\, \xi_i \xi_j \geqslant \alpha\, |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n, \alpha > 0, \qquad \text{for a.e.} \quad x \in \Omega.$$

The solutions are assumed to lie in $H^1(\Omega) = W^{1,2}(\Omega)$. A fundamental result, whose proof relies on a sophisticated application of the above principles, asserts that every solution $u \in H^1(\Omega)$ of $Lu = 0$ is continuous, and more precisely belongs to some $C^{0,\alpha}$. A similar conclusion was derived independently by J. Nash [Na2] for the corresponding parabolic equation.

As we have already mentioned in Section 7 these estimates are the first and basic steps in solving Hilbert's 19th problem, i.e., in proving that the variational problem associated with the functional

$$\int_{\Omega} F(x, u, \operatorname{grad} u)$$

with $u = \varphi$ on $\partial\Omega$ has a smooth minimum provided F is smooth and the corresponding Euler–Lagrange equation is uniformly elliptic. This result completed a long lasting effort to establish regularity of weak solutions for *scalar* problems, i.e., where u is a real valued function.

In a number of important physical and geometrical situations u is not a scalar but a *vector* and the corresponding Euler–Lagrange equation is a *system.* The question arose naturally whether the previous theory extends to systems. In 1968 E. DeGiorgi [Dg2] constructed a surprising counterexample of a second order linear elliptic system $Lu = 0$ where the solution has the form $x/|x|^{\alpha}$, $\alpha > 1$, and thus is not continuous. DeGiorgi [Dg2] and independently Mazya [Maz1] also found a scalar equation $Lu = 0$ with L linear elliptic of order 4 for which the solution is unbounded. For nonlinear variational systems Giusti and Miranda [G-M] constructed an example involving a smooth F, where the minimizer has the form $x/|x|$. This ruled out any transparent theory of regularity for solutions of systems. In 2-d such regularity does hold as was established by C. B. Morrey [Mor2] in the 1940's. In higher dimensions *partial regularity* was established starting with the works of Almgren, DeGiorgi, Federer, Giusti, M. Miranda and Morrey in the late 1960's, showing that the singular set of a solution is small in the sense of appropriate Hausdorff measure. There has been renewed interest in partial regularity during the 1980's, motivated in particular by nonlinear elasticity and harmonic maps, with contributions by a number of authors including Evans, Gariepy, Giaquinta, Giusti, Hildebrandt, G. Modica, Necas; see e.g. the books of Giaquinta [Gia1,2] and Necas [Nec]. A remarkable result of Schoen–Uhlenbeck [S-U] asserts that the singular set of a minimizing harmonic map in $\mathbb{R}^n$ has Hausdorff dimension $\leqslant n-3$. For example, the singular set in 3-d consists of *isolated points*; moreover Brezis, Coron and Lieb [B-C-L] have shown that every singularity has the form $x/|x|$. This is consistent with the observation of *point defects* in some physical problems (e.g. in the theory of materials, such as liquid crystals). *Line singularities* (in 3-d), e.g. Ginzburg–Landau vortices, occurring in superconductors and superfluids, have recently been investigated by Bethuel, Brezis and Hélein [B-B-H].

Another sophisticated class of applications of the maximum principle, in conjunction with the method of moving planes of A. D. Alexandrov, consists of establishing geometric properties of the solutions. This program was initiated by J. Serrin [Ser2] in 1972 and pursued among others by B. Gidas, W. M. Ni and L. Nirenberg [G-N-N] who established that any positive solution of

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega = \text{a ball} \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

has radial symmetry. Here f can be very general—a Lipschitz condition suffices. Again, little is known about the analogue for systems.

As the above observations indicate, the validity of the maximum principle is restricted to *second order*, *scalar*, elliptic operators and does not extend in any natural way to systems of second order operators or to higher order scalar equations. This creates a discontinuity in the type of conclusions for those two cases.

For questions discussed in this Section we refer the reader to the books of Protter–Weinberger [P-W], Stampacchia [Sta], Gilbarg–Trudinger [G-T], Giaquinta [Gia1,2], Ladyzhenskaya–Uraltseva [L-U], Morrey [Mor3] and Necas [Nec].

20. NONLINEAR EQUATIONS OF EVOLUTION: FLUID FLOWS AND GAS DYNAMICS

A wide variety of problems of the greatest importance in physics and engineering are formulated in terms of nonlinear equations of evolution. The most general form of such equations is given by

$$\frac{\partial u}{\partial t} = Au$$

where the nonlinear operator A and the space of functions on which it acts are specified by the nature of the problem.

Historically, the equations which have received the most intensive study, particularly from the point of view of constructing a rigorous mathematical theory, arise in the description of incompressible fluid flows: the Navier–Stokes equation (4) and the Euler equation (5). In this case the possibility of a *blow-up* of the solution, i.e., a time T^* in which either the solution or some of its derivatives become infinite somewhere, has been associated by J. Leray and others with the physical phenomenon of *turbulence*, one of the most significant macroscopic problem in physics. The study of possible singularities for Navier–Stokes in 3-d which was begun by J. Leray in 1933 (see Section 11) has been carried further by V. Scheffer (1977) and L. Caffarelli, R. Kohn and L. Nirenberg [C-K-N] in 1982 to exclude for example a line of singularities in space-time. Whether singular points exist at all in the 3-d Navier–Stokes and Euler equations is still a major open problem.

In 2-d, global existence and regularity have been proved (see Section 11) and the question of the description of the behavior of the solution as $t \to \infty$ has focused on concepts from dynamical systems, in particular the study of *attractors*. It was suggested by D. Ruelle and F. Takens in a much discussed paper that the phenomenon of turbulence might be derived from the possible existence of a complicated attractor. Such attractors and their chaotic behavior have been studied in great detail for some finite dimensional systems by S. Smale and his school. In 1963 the meteorologist E. Lorenz discovered a simple system of three ordinary differential equations for which numerical computations indicated a complicated asymptotic structure as $t \to \infty$. Much of the recent research in dissipative nonlinear equations of evolution having global solutions has focused on the question of reducing the study to the solution as $t \to \infty$ to a finite dimensional situation, especially through the works in the 1980's of Babin and Visik, Foias and Temam, and Ladyzhenskaya; see e.g. the books of Temam [Te3] and Ladyzhenskaya [L2]. A strong stimulus for these investigations has been provided by the discovery of the Feigenbaum cascade describing some universal phenomena in the iteration of mappings. Such cascades have also been discovered experimentally in certain investigations of fluid flows.

Besides the Navier–Stokes equation, other equations have been studied from the point of view of their attractors, such as the Kuramoto–Sivashinski equation

$$\frac{\partial u}{\partial t} + \nu \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0, \qquad \nu > 0$$

arising in combustion theory, or the Cahn–Hilliard equation

$$\frac{\partial u}{\partial t} + \nu \Delta^2 u - \Delta(u^3 - \alpha u) = 0$$

where $\nu > 0$, arising in phase transitions.

Still in the context of global solutions of fluid flows, a different phenomenon appears in the Korteweg–DeVries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

which describes waves in shallow channels. Here, one has the phenomenon of *soliton* solutions, first noticed by J. S. Russell in 1834. These are solutions which preserve their shape indefinitely and even interact with other such solutions without losing their individuality. A theory of soliton solutions for *KdV* was initiated in the mid 1960's by M. Kruskal and his collaborators (see [Z-K] and [G-G-K-M]). The main idea is to introduce a change of variable, based on inverse scattering for the 1-d Schrödinger operator (the Gelfand–Levitan theory, see Section 18) which makes the problem linear and explicitly solvable. Shortly afterwards P. Lax [La4] formalized this method and introduced the so-called "Lax pair". In 1971 V. E. Zakharov and A. B. Shabat recognized that the Lax formalism is not restricted to the *KdV* equation, but can also be used for the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+k\,|u|^2\,u=0, \qquad k\in\mathbb{R}.$$

This has given rise to a broad attack in the 1970's on the study of other nonlinear equations of evolution in a wide variety of physical and engineering contexts, for example the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+\sin u=0$$

and the Toda lattice. The systems involved have the property of being *completely integrable* and have a large or infinite number of invariants of motion, something which is not the case for dynamical systems or equations of evolution in general. The generalization of this theory has led to an extremely fruitful interaction in the 1980's between PDE's and areas of mathematics and physics like algebraic geometry, group theory (quantum groups) topology (connections between knot theory, Jones polynomials, and integrable systems) and quantum gravity. We refer to the books [New], [F-T] and [F-Z].

Going back to the original motivation in fluid mechanics, long waves in nonlinear dispersive systems have been studied by a number of authors, e.g. T. B. Benjamin, J. L. Bona and J. J. Mahony [B-B-M].

Another direction of investigation which arises in gas dynamics and compressible flows is the theory of nonlinear *conservation laws* and *shock waves*. For a scalar equation they have the form

$$\frac{\partial u}{\partial t}+\sum_i\frac{\partial}{\partial x_i}\,\varphi_i(u)=0, \tag{12}$$

where the functions φ_i are smooth. A special case is Burger's equation already mentioned in Section 11. Solutions corresponding to special initial conditions were constructed by Riemann (1858). The general theory of such equations was started in 1950 by E. Hopf [Hop3] and continued in 1957 by O. Oleinik [O1] and P. Lax [La3] (see also the important programmatic paper of Gelfand [Ge]). Shock waves, i.e., solutions with jump discontinuities, appear as a natural and inevitable structure of the problem. For most smooth initial data there is no global smooth solution. For a given initial condition a plenitude of weak solutions (in the sense of distributions, see Section 13) exist. A selection mechanism, which singles out the physically interesting solution is the *entropy condition.* This special solution can also be characterized as the limit for small viscosity ν of the solution of

$$\frac{\partial u}{\partial t} - \nu \, \Delta u + \sum_i \frac{\partial}{\partial x_i} \varphi_i(u) = 0$$

which admits globally defined smooth solutions for each $\nu > 0$. A fairly complete existence and uniqueness theory is available in this situation. More precisely any two entropy solutions u, v satisfy

$$\int |u(x, t) - v(x, t)| \, dx \leqslant \int |u(x, 0) - v(x, 0)| \, dx \qquad \forall t \geqslant 0 \tag{13}$$

This was first established by S. Kruzhkov in 1970 and then revisited by M. Crandall in the framework of nonlinear semigroups (see Section 21).

No broad theory yet exists for systems of conservation laws. Despite an important advance made in 1965 by J. Glimm [Gl], in 1970 by J. Glimm–P. Lax [Gl-La], and in the 1970's and 80's by C. Dafermos, J. Smoller, R. DiPerna and T. P. Liu, many difficult questions still remain open. We refer to the book of J. Smoller [Sm] for a survey of the field up to 1983; the book of Courant and Friedrichs [C-F] provides a good description of results obtained during the first half of this century.

A nonlinear, physically fundamental, equation of evolution is the celebrated Boltzman equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \xi \cdot \operatorname{grad}_x u = Q(u, u) & x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad t > 0 \\ u(x, \xi, 0) = u_0(x, \xi) \end{cases}$$

where u is a function of (x, ξ, t) and Q is a quadratic collision term. Existence (in an appropriate weak sense) has been established in 1983 by R. DiPerna and P. L. Lions [D-L]. As in the 3-d Navier–Stokes equation the questions of global regularity and uniqueness of the weak solution remain open.

The actual existence of blow-up solutions in some nonlinear equations of evolution (both parabolic, hyperbolic and Schrödinger) is easy to verify as was observed e.g. by H. Fujita [Fu] in 1966 for the equation

$$u_t - \Delta u = u^p, \qquad p > 1.$$

The blow-up mechanism for semilinear heat equations has been carefully investigated in the 1980's and 1990's. In some cases there is a *blow-up profile*, i.e., $\lim_{t \uparrow T^*} u(x, t)$ exists and is finite, except for x in a small set, where T^* denotes the blow-up time. A number of authors have analyzed the behavior of $u(x, t)$ as $t \uparrow T^*$, e.g. F. Weissler [Wei], Y. Giga and R. Kohn, M. A. Herrero and J. J. L. Velasquez, A. Friedman and J. B. McLeod. Others have investigated the delicate question whether the solution can be extended in a natural way *beyond blow-up time*, e.g. P. Baras and L. Cohen, V. A. Galaktionov and J. L. Vazquez; we refer to the book of [B-C].

Similar questions are currently studied for semilinear wave and Schrödinger equations

$$u_{tt} - \Delta u \pm |u|^{p-1} u = 0$$

$$iu_t - \Delta u \pm |u|^{p-1} u = 0.$$

Important advances have been made beginning with K. Jorgens and continued by many writers including Ginibre, Velo, Brenner, Grillakis, Struwe, Klainerman, Kenig, Ponce and Bourgain. Difficult problems remain open. The characterization of situations where blow-up actually occurs is one of the important questions of the theory of nonlinear equations of evolution. An interesting direction of current research is the discovery by F. John, S. Klainerman and others, that for $n \geqslant 4$, the nonlinear wave equation

$$u_{tt} - \Delta u = F(x, t, u, u_t, u_x), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

where F starts with quadratic terms at 0, has global solutions for small initial data. When $n = 3$ a similar conclusion fails but special conditions on F (the so-called null conditions) give rise to global solutions for small initial data. This is in sharp contrast with the case $n = 1$ where singularities develop in finite time for arbitrarily small (but not identically zero) initial data as in the case of shock waves described above. Christodoulou and Klainerman [C-K] have partially extended this analysis to other hyperbolic equations such as Einstein's equation in general relativity and Yang–Mills equation.

Special solutions of various nonlinear evolution equations, called travelling waves (or fronts), have the form $u(x, t) = \varphi(x - ct)$ in $1 - d$ or $u(x, t) = \varphi(x_1 - ct, x')$ with $x = (x_1, x')$ in general. They occur in a variety of applications, such as population genetics, combustion and other propagation phenomena. This subject has been extensively studied since the pioneering paper of A. Kolmogorov, I. G. Petrovsky and N. S. Piskunov [K-P-P]; see e.g. D. G. Aronson and H. F. Weinberger [A-W] and H. Berestycki and L. Nirenberg [Be-N].

21. NONLINEAR PDE'S AND NONLINEAR FUNCTIONAL ANALYSIS

The method of *successive approximation* extensively studied since Picard in the 1880's (see Section 5) was stated in an elegant and general setting by S. Banach in 1922. It asserts that in a complete metric space X a mapping $f: X \to X$ which satisfies

$$d(f(x), f(y)) \leqslant kd(x, y) \qquad \forall x, y \in X, k < 1$$

has a unique fixed point x_0 given by $x_0 = \lim_{n \to \infty} f^n(a)$ for any initial point $a \in X$.

A consequence of this is the inverse function theorem which states that if F maps a neighborhood U of $u_0 \in X$ into Y, where X and Y are Banach spaces and F is C^1 on U with $L = F'(u_0)$ one-to-one and onto Y. Then the equation $F(u) = f$ has a unique solution in a neighborhood of u_0, for every f in a neighborhood of $f_0 = F(u_0)$.

A program of extending such results when $F'(u_0)$ is not invertible, called *bifurcation theory*, originated in the work of A. M. Lyapunov (1906) and E. Schmidt (1908) to deal with problems first posed in 1885 by H. Poincaré in connection with astrophysics. The typical situation concerns a one parameter family of maps $F_\lambda(u)$ with $\lambda \in \mathbb{R}$ where $F_\lambda(0) = 0$, $\forall \lambda$, and $F_0(u)$ has a derivative at 0, $F_0'(0) = L$ which has a nullspace of dimension one and a closed range of codimension one. Under simple hypotheses one establishes the existence for λ *near* 0 of a branch of nonzero solutions $u(\lambda)$ of $F_\lambda(u(\lambda)) = 0$. Such results have proved to be enormously useful in a wide variety of applications in physics and engineering such as buckling problems in elasticity, thermal convection and rotating fluids. The extension to the case where the dimension of the nullspace of L is greater than one was carried through in the early 1950's by M. A. Krasnoselskii and his school using variational and topological methods (see [Kra]). The most definitive result on the existence of *global* branches, i.e., λ running through $\mathbb{R}$, was obtained in 1971 by P. Rabinowitz [Ra1] applying the degree theory of Leray–Schauder (see Section 9). It asserts that under compactness hypotheses each branch either extends to infinity (in $X \times \mathbb{R}$) or runs into another bifurcation point.

This illustrates a persistent division of results between local and global. Local results are often obtained by a perturbation argument from a linear situation by some variant of successive approximations, while global results usually require some sophisticated tools, such as variational or topological arguments, often combined with a priori estimates.

The most general and sophisticated form of the perturbation argument was devised in 1956 by J. F. Nash [Na1] in his proof of the existence of C^∞ isometric imbeddings of Riemannian manifolds in Euclidean space. In this case one has a mapping F of the space $X = C^\infty(M)$ into another space $Y = C^\infty(N)$ such that $F'(u_0)$ has a continuous inverse L which does not completely recover the regularity lost under the action of the differential operator F. Since X and Y are not complete normed spaces the inverse function theorem does not apply. J. F. Nash devised an argument for obtaining a local inverse for F by combining iterations of L, F and smoothing operators. This argument was modified in 1966 by J. Moser [Mos2] and applied to the problem of establishing the C^∞ analogue of the results of Kolmogorov and Arnold in the analytic case on the existence of quasi-periodic orbits as perturbations of periodic orbits in Hamiltonian systems (like those of celestial mechanics). A comprehensive survey may be found in the paper by R. Hamilton [Ham1].

Concerning *global* results one of the key advances was the Leray–Schauder degree theory (see Section 9). Another very powerful approach was the introduction of topological tools into the study of variational problems. Though this approach was foreshadowed by Poincaré and Birkhoff the force of these ideas was realized in the late 1920's and early 1930's in the works of Ljusternik and Schnirelman [Lj-Sc] and Morse [Mrs1,2]. In particular, Ljusternik and Schnirelman, in the case of variational problems on finite dimensional manifolds, gave a lower bound for the number of critical points in terms of topological invariants, e.g. the Ljusternik–Schnirelman category. Morse's theory for nondegenerate functions Φ gives a finer classification of critical points in terms of the quadratic forms associated with $\Phi''(u_0)$. Morse applied this method to the study of a classical problem posed by Poincaré: the existence of infinitely many geodesics on an elliptic energy surface. In the early 1960's the ideas of Morse theory were put into the framework of differential topology on *infinite dimensional* manifolds M by R. Palais and S. Smale [P-S] who replaced the finite dimensionality assumption in the original theory by an assumption of compactness type: condition (C) (or (PS)) which states that if one has a sequence (u_n) in M for which $|\Phi(u_n)|$ is bounded and $\Phi'(u_n) \to 0$ then (u_n) is relatively compact in M. This generalization opened the door for a wide range of applications in PDE's where the basic framework consists of infinite dimensional spaces (or manifolds) of functions. Later, other problems falling outside the Palais–Smale framework were considered, using sophisticated modifications of the Morse and Ljusternik–Schnirelman theory. Some of these are of great importance in geometry and physics (see Section 24).

In particular, it is not always possible to minimize a nonnegative continuous function Φ on a complete metric space. A useful principle due I. Ekeland [Ek] asserts e.g. that for a nonnegative C^1 function on a Banach space there is always a minimizing sequence (u_n) such that $\Phi'(u_n) \to 0$.

One specific method, which extends ideas already present in Poincaré and Birkhoff and is simpler than the theories mentioned above, is the well-known "mountain pass" lemma of A. Ambrosetti and P. Rabinowitz [A-R]. It asserts that if Φ is a C^1 function on a Banach space X, satisfies condition (C) above and the following geometric condition:

$$\begin{cases} \Phi(0)=0, & \Phi(v) \geqslant \alpha > 0 \quad \text{for all} \quad v \in X \quad \text{with} \quad \|v\| = R \quad \text{and} \\ \Phi(v_0) \leqslant 0 & \text{for some} \quad v_0 \in X \quad \text{with} \quad \|v_0\| > R, \end{cases}$$

then there exists a nontrivial critical point u of Φ, i.e., $\Phi'(u)=0$ and $\Phi(u) \geqslant \alpha$.

A simple consequence is the existence of solutions for problems of the form

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where p is subcritical, i.e., $1 < p < (n+2)/(n-2)$.

The critical points obtained by the topological methods described above are generally nonstable critical points which are neither maxima nor minima (they are sometimes called saddle points). For a survey of these questions we refer to the books [Ra2], [Ch], [M-W], [Str2] and [Ze].

The simplest method in the calculus of variations is the *direct method*: one looks for a *minimum* of Φ which is to be obtained as limit (in some appropriate sense) of a minimizing sequence. As we have already mentioned in Section 6 this raises serious issues about the convergence of minimizing sequences. Though the domain of the functional Φ is almost never compact in the infinite dimensional case, one uses other properties of the functional to enforce convergence of the minimizing sequence. An important topology in which such arguments can be carried out is the *weak topology* on a reflexible Banach space (such as $L^p(\Omega)$ or $W^{m,p}(\Omega)$ for $1 < p < \infty$) in which the unit ball is weakly compact. The decisive property for the functional Φ is its weak lower semicontinuity (l.s.c.), i.e., if u_k tends to u weakly (denoted $u_k \rightharpoonup u$), then $\Phi(u) \leqslant \liminf_{k \to \infty} \Phi(u_k)$.

A basic sufficient condition for a continuous Φ to be weakly l.s.c. is that Φ is convex, i.e.,

$$\Phi(tu + (1-t)\, v) \leqslant t\Phi(u) + (1-t)\, \Phi(v) \qquad \forall u, v \in X, \quad \forall t \in [0, 1].$$

If one expresses convexity in terms of the derivative Φ' of Φ it becomes

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle \geqslant 0 \qquad \forall u, v \in X \tag{14}$$

where $\Phi'(u)$ is considered as an element on the dual space X^* of the space X on which Φ is defined. Inequality (14) leads to the introduction of mappings A from X into X^* satisfying

$$\langle Au - Av, u - v \rangle \geqslant 0 \qquad \forall u, v \in X$$

called *monotone mappings.* Such inequalities can even be applied to multivalued mappings which are not necessarily defined at each point of X. The study of monotone mappings in Hilbert spaces was begun in 1962 by G. Minty [Min], who proved that if A is maximal monotone, i.e., A cannot be extended to a larger (multivalued) monotone mapping, then $A + I$ is surjective. This is a nonlinear generalization of the Lax–Milgram lemma (see Section 14).

An important domain of applications of this concept to PDE's is the class of elliptic differential operators in the generalized divergence form, introduced in 1963 by M. I. Visik [Vi2]:

$$Au = \sum_{|\alpha| \leqslant m} (-1)^{\alpha} D^{\alpha} A_{\alpha}(x, u, Du, ..., D^m u)$$

where A_α is a general nonlinear function. If there is a function f such that $A_\alpha = \partial f / \partial p_\alpha$, where p_α corresponds to D^α, then A is the derivative of Φ defined by

$$\Phi(u) = \int_\Omega f(x, u, Du, ..., D^m u)$$

i.e., A is the Euler–Lagrange differential operator associated to the functional Φ.

In general, if the functions A_α satisfy the algebraic monotonicity conditions

$$\sum_{|\alpha| \leqslant m} (A_\alpha(x, \xi) - A_\alpha(x, \xi^\#), \xi_\alpha - \xi^\#_\alpha) \geqslant 0, \qquad \forall \xi, \xi^\#, \tag{15}$$

where $\xi = (\xi_\alpha)$ is the m-jet of u, then under suitable growth conditions on A_α, the operator A maps $X = W_0^{m,p}(\Omega)$ into $X^* = W^{-m,p'}(\Omega)$ and is monotone. If in addition

$$\sum_{|\alpha| \leqslant m} (A_\alpha(x, \xi), \xi_\alpha) \geqslant c\, |\xi|^p \qquad \forall \xi, \qquad \text{with} \quad c > 0,$$

then A is *coercive*, i.e.,

$$\frac{\langle Au, u \rangle}{\|u\|_{W^{m,p}}} \to \infty \qquad \text{as} \quad \|u\|_{W^{m,p}} \to \infty.$$

In 1963 F. Browder [Bro5] showed that monotonicity, continuity and coerciveness imply surjectivity in a reflexive Banach space. In 1965, J. Leray and J. L. Lions ([L-L] and [Lio2]) weakened the hypothesis (15) to a monotonicity condition involving only the top order terms. In 1968 H. Brezis [Bre1] subsumed their results under a more general theory, that of *pseudo-monotone* operators. An operator A from X into X^* is pseudo-monotone if for any weakly convergent sequence $u_k \rightharpoonup u$ such that $\limsup_{k\to\infty} \langle Au_k, u_k - u\rangle \leqslant 0$ one has

$$Au_k \rightharpoonup Au \qquad \text{and} \qquad \langle Au_k, u_k - u\rangle \to 0.$$

(See also the later paper of Landes and Mustonen [Lan-M].)

Such a definition illustrates the principle that nonlinear mappings are, in general, *not continuous in the weak topology* (unlike the bounded linear operators). This is a *major source of complications* in the study of nonlinear PDE's. More recent treatments of this theme can be found in the survey by Evans [Ev2] and in [Dac]. The topics discussed include weak lower semicontinuity for quasi-convex functionals as defined by Morrey [Mor3] and applied to nonlinear elasticity by Ball [Bal], (see also Antman [Ant]). Another topic is compensated compactness as defined by L. Tartar and F. Murat, and applied to nonlinear hyperbolic problems by DiPerna [DiP].

In addition, the degree theory of Leray–Schauder has been extended to the framework of operators of monotone type by F. Browder and W. Petryshyn [B-P], F. Browder [Bro7] and I. Skrypnik [Sk]. Though there has been extensive activity on degree theory for noncompact operators in the last two decades, we refer only to another area relevant to PDE's. This is the theory of nonlinear mappings of Fredholm type (proper mappings of index zero) sketched by R. Cacciopoli [Ca2] in 1936 and by S. Smale [Sma] in 1965 and carried through in detail by K. Elworthy and A. Tromba [E-T].

A different area of applications for monotone operators is their role as the infinitesimal generators of *nonlinear semi-groups of contractions*. If one has the abstract differential equation

$$\begin{cases} \dfrac{du}{dt} = Au, & t \geqslant 0, \\[2ex] u(0) = u_0 \end{cases}$$

in a Hilbert space H, the transition operator $U(t)\colon u_0 \mapsto u(t)$ satisfies the contraction property

$$\|U(t)\,u_0 - U(t)v_0\| \leqslant \|u_0 - v_0\|, \qquad \forall u_0, v_0, \quad \forall t \geqslant 0,$$

if and only if $-A$ is monotone. A fairly complete generalization of the Hille–Yosida theory in Hilbert spaces has been developed by many authors including F. Browder, T. Kato, Y. Komura, M. Crandall, A. Pazy and H. Brezis. The principal result asserts that there is a one-to-one correspondence between continuous semi-groups of contractions and maximal monotone operators. We refer to the books of H. Brezis [Bre2], F. Browder [Bro6] and V. Barbu [Bar].

In the case of Banach space the theory involves the notion of m-accretive operator, i.e., $A\colon D(A) \subset X \to X$ satisfies $R_\lambda = (I + \lambda A)^{-1}$ is well defined on all of X for all $\lambda > 0$ and is a contraction. A noteworthy result from 1971 by M. Crandall and T. Liggett [C-L] asserts that, in a general Banach space X, every m-accretive operator generates a contraction semi-group in a suitable generalized sense.

The results that we have mentioned, as well as recent developments, have the important property of moving the study of nonlinear problems beyond the framework of compactness required in the Leray–Schauder theory or the Palais–Smale condition. This remains a fundamental question in the future development of nonlinear functional analysis; see e.g. Section 24 and [Bre3].

22. FREE BOUNDARY VALUE PROBLEMS: VARIATIONAL INEQUALITIES

Up to this point we have considered BVP's for equations, linear or nonlinear, on a *given domain*. A class of problems of importance in many applications concerns free boundary problems. The *domain* on which the solution is defined is *part of the unknown* of the problem. The boundary data, in turn, are overdetermined in the classical sense.

One of the simplest example is the following. Let $f(x)$ be a given function on a given domain Ω in $\mathbb{R}^n$. Find a subdomain D and a function u on D satisfying

$$\begin{aligned} -\Delta u &= f && \text{in } D, \\ u &= 0 && \text{on } \partial D, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } (\partial D) \cap \Omega, \end{aligned}$$

where ν denotes the normal to ∂D. Similar problems occur in evolution equations. Such problems arise in fluid mechanics, e.g. the Stefan problem for a mixture of ice and water, filtration through a porous dam, wakes and cavities.

There are two levels of basic difficulties:

(1) Establishing *existence and uniqueness* of the solution (u, D),

(2) Establishing *regularity* properties of (u, D).

Methods of studying such problems have been relatively ad hoc. One fairly systematic approach to the existence problem lies through the theory of *variational inequalities* extensively studied in the late 1960's and 1970's. This arises as an extension of the Dirichlet principle. Here one minimizes the Dirichlet integral

$$\frac{1}{2}\int_{\Omega} |\mathrm{grad}\, u|^2 - \int_{\Omega} fu$$

over the class of testing functions in a convex set K, e.g. $u \geqslant 0$ in Ω. The minimizer exists, is unique and satisfies

$$-\Delta u = f \qquad \text{in} \quad D = \{x \in \Omega;\, u(x) > 0\}.$$

Moreover it can be proved (see H. Lewy and G. Stampacchia [Le-St], H. Brezis and G. Stampacchia [B-S]) that $u \in C^{1,\alpha}(\Omega)$ and (16) holds.

Reduction of free boundary value problems to variational inequalities can often be sophisticated, see e.g. C. Baiocchi [Ba]. The regularity of the free boundary has been studied by numerous authors including H. Lewy, D. Kinderlehrer, L. Nirenberg and L. Caffarelli. The interested reader will find an extensive presentation of free boundary value problems in the books of Baiocchi and Capelo [Ba-Ca], A. Friedman [Fr], and Kinderlehrer and Stampacchia [K-S].

23. QUASILINEAR AND FULLY NONLINEAR ELLIPTIC EQUATIONS

As we have already mentioned (see Sections 7 and 9) one of the key tools for proving existence of solutions of BVP consists of finding a priori estimates for the solutions. This was begun in the work of S. Bernstein and was continued for almost a century. We recall that a quasilinear elliptic equation of second order is an equation of the form

$$\sum_{i,j} a_{ij}(x, u, Du) \frac{\partial^2}{\partial x_i \, \partial x_j} = f(x, u, Du) \tag{17}$$

where the (a_{ij}) are elliptic. On the other hand, a fully nonlinear elliptic equation is one of the form

$$F(x, u, Du, D^2u) = 0 \tag{18}$$

where F is elliptic at a solution u provided the linearized equation at u is elliptic, i.e.,

$$\sum_{i,j} \frac{\partial F}{\partial p_{ij}}(x, u, Du, D^2u)\, \xi_i \xi_j \geqslant c\, |\xi|^2 \qquad \forall \xi \in \mathbb{R}^n, \quad c > 0,$$

where (p_{ij}) corresponds to the Hessian matrix (D^2u).

As we have noted in Section 19 regularity results involving a priori estimates in $C^{2,\alpha}$ are essentially limited to two cases:

— a single equation for any dimension $n \geqslant 2$

— fairly general systems for $n = 2$.

The earlier development of a priori estimates in $C^{2,\alpha}$ and in C^k, $k \geqslant 3$, was carried through first in the quasilinear case. The pioneering work of S. Bernstein was continued by many authors including J. Leray, J. Schauder, C. Morrey, L. Bers, L. Nirenberg, O. Ladyzhenskaya, N. Uraltseva, and J. Serrin. A complete theory, developed by O. Ladyzhenskaya–N. Uraltseva [L-U], provides interior estimates of solutions of (17), as well as estimates up to the boundary for the same equation with a boundary condition. For a detailed account see [L-U], [G-T] and [Ser1]. For a broad survey of the theory of singularities for quasilinear equations we refer to the book of L. Veron [Ve] which covers e.g. the early work of L. Bers and J. Serrin.

More recently, corresponding results have been established for a broad class of fully nonlinear equations (18), notably including the Monge–Ampère equation

$$\det(D^2u) = f(x, u, Du) \tag{19}$$

of great importance in geometrical problems, and the Hamilton–Jacobi–Bellman equation

$$\operatorname*{Sup}_{i \in I}\{A_i u - f_i\} = 0 \tag{20}$$

where $(A_i)_{i \in I}$ are a family of linear second order elliptic operator. The latter equation appears in stochastic control theory.

In 2-d, a complete theory of a priori estimates for fully nonlinear equations (18) was derived in 1953 by L. Nirenberg [Ni1] using techniques developed earlier by C. Morrey [Mor1].

In 3-d and higher dimensions, the general problem (18) is still open, namely, to find $C^{2,\alpha}$ a priori estimates for C^2 solutions in the fully nonlinear case. Once one obtains $C^{2,\alpha}$ estimates, standard techniques yield C^∞ (or even analytic) regularity provided F is C^∞ (or analytic). The earliest results for the Monge–Ampère equation are due to A. D. Aleksandrov,

A. V. Pogorelov, I. Bakelman, S. Y. Cheng and S. T. Yau, and for the Hamilton–Jacobi–Bellman equation to N. V. Krylov using techniques of probability (via a representation formula for the solution). A further treatment of the equation (20) was carried through in the early 1980's by a number of authors including H. Brezis, L. C. Evans, A. Friedman and P. L. Lions using purely PDE methods. Complete estimates up to the boundary for the Monge–Ampère equation (19) are due to Caffarelli, Nirenberg and Spruck [C-N-S] and to Krylov [Kry2]. For the Hamilton–Jacobi–Bellman equation, the final result, $C^{2,\alpha}$ estimates up to the boundary, was obtained by N. V. Krylov [Kry1,2,3] in the mid-1980's.

For the more general class of fully nonlinear equations (18), the most striking result was obtained independently by L. C. Evans [Ev] and N. V. Krylov [Kry1] who proved $C^{2,\alpha}$ estimates if in addition F is concave in (D^2u). On these questions we refer to the books [Au], [G-T], [Kry3] and [C-C].

A fundamental new tool in this context is the discovery in 1980 by Krylov and Safonov [Kr-Sa] that Harnack's inequality and $C^{0,\alpha}$ estimates hold for second order linear elliptic equations with bounded measurable coefficients in *nondivergence form* (this is the analogue of the DeGiorgi–Nash estimates for equations in divergence form).

24. PDE'S AND DIFFERENTIAL GEOMETRY

In the past decades there has been a powerful tendency to merge geometry and theoretical physics, embodied in such areas as general relativity, Yang–Mills equations and other gauge theories, and most recently in super-symmetric string theories. In all these contexts, as well as in the more classical geometrical applications, the use of PDE's takes place on two different levels: the linear and the nonlinear.

Beginning with classical potential theory and its application to the study of Riemann surfaces and algebraic curves, the development of the theory of linear PDE's, as we have mentioned in Section 5, went hand in hand with the rise of classical function theory and algebraic geometry. This was extended in the proof of Hodge's theorem to manifolds of dimension greater than two, especially in the work of Kodaira and Spencer, made possible a significant extension of the earlier results to the higher dimensional case. Even more recently, the PDE approach to holomorphic functions of one variable in terms of harmonic functions and the Dirichlet problem has been extended to holomorphic functions of several variables using the $\bar{\partial}$-Neumann problem first solved by J. J. Kohn [Koh]. This has given rise to an extensive analysis of the structure of holomorphic functions in several variables, by reducing such problems to subelliptic BVP's.

When one turns to nonlinear equations, the most conspicuous is the *Plateau problem* for minimal surfaces; it consists of finding a surface with least area spanning a given contour in $\mathbb{R}^3$. In non-parametric form such a surface is described by a function u satisfying the minimal surface equation

$$\operatorname{div}\left(\frac{\operatorname{grad} u}{(1+|\operatorname{grad} u|^2)^{1/2}}\right)=0 \tag{21}$$

which is the Euler–Lagrange equation for the area functional

$$\int (1+|\operatorname{grad} u|^2)^{1/2}.$$

Equation (21) is nonlinear, but the successful solution of Plateau's problem in 1931 by J. Douglas [Do] used a "linearization" of the problem in terms of the theory of holomorphic functions. This is an example of a process which, when it works, may achieve dramatic results: namely, an ad-hoc mechanism, usually of ingenious form, for transforming the nonlinear problem into a linear one. A second example is the use of linear scattering theory by Kruskal and his collaborators to reduce the study of the $K\,dV$ equation to spectral properties of the Schrödinger operator (see Section 20). A third striking example was the linearization in certain cases by Atlyah and Ward of the instanton theory for the Yang–Mills equations using Penrose's theory of twistors. In most cases there seems to be no possibility of reducing the nonlinear equations which arise in various geometric or physical contexts to linear problems. Hence one must apply the full strength of the nonlinear theory with all the technical difficulties that it incurs, e.g. a priori estimates and delicate analytical inequalities. Let us describe briefly some of the most prominent nonlinear PDE's arising in geometry:

(1) The *minimal surface equation* (21) in n dimensions which was studied, using a combination of techniques from geometric measure theory and PDE's, by many authors including Federer, Fleming, Reifenberg, DeGiorgi, Bombieri, Giusti, M. Miranda, Finn, J. Nitsche, Jenkins, Serrin, Almgren, Allard, Simons and others. A central result of minimal surface theory, from the point of view of PDE's, asserts that an entire solution (i.e., a solution on all of $\mathbb{R}^n$) of the minimal surface equation (21) is linear if $n \leqslant 7$. This was first established by S. Bernstein in 1916 when $n=2$. A celebrated counterexample was produced in 1969 by Bombieri, De Giorgi and Giusti [B-D-G] in $\mathbb{R}^8$. An important consequence, via a blow-up analysis, is the regularity of minimal hypersurfaces in dimension $\leqslant 7$ and an estimate for the dimension of the singular set in higher dimension (its Hausdorff dimension cannot exceed $n-8$). The interested reader may consult

the books by E. Giusti [Gi] and V. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab [D-H-K-W] and J. Nitsche [Nit].

(2) *The Yamabe equation.* In 1960 Yamabe claimed to have proved that for a compact Riemannian manifold (M, g) of dimension $n \geqslant 3$ there is a metric g', conformal to the original metric g, for which the scalar curvature is constant. This is equivalent to finding a positive function u satisfying the nonlinear elliptic equation

$$-4\frac{(n-1)}{(n-2)}\Delta_g u + Ru = Ku^{(n+2)/(n-2)} \tag{21}$$

where R is the scalar curvature of the metric g and K is a constant.

As was pointed out by N. Trudinger, the original argument contained a major gap which attracted much attention. The positive result was obtained by T. Aubin in 1975 for $n \geqslant 6$ (see [Au]), making extensive use of the theory of Sobolev spaces (see Section 12) and the best constants in the Sobolev imbedding. The most important missing cases were treated by R. Schoen [Sco] in 1984 with the help of the positive mass conjecture established by R. Schoen and S. T. Yau.

A more general form of the Yamabe problem, dealing with equation (21) where $K = K(x)$ is a function, has been investigated since the mid-1970's by numerous authors, e.g. Kazdan and Warner, Escobar and Schoen, Bourguignon and Ezin, Bahri and Coron, S. Y. Chang and P. Yang, and others.

This type of problem is particularly interesting since it can be formulated as a variational problem on the Sobolev space H^1 for a functional Φ which does not satisfy the Palais–Smale condition (C) (see Section 21); it is a borderline case for the Sobolev imbedding and this may create an obstruction to existence as was first pointed out by Pohozaev [Poh]. This *lack of compactness*, caused by scale and conformal invariance, is connected with the "bubbling" phenomenon originally identified in 1981 by Sacks and Uhlenbeck [Sa-Uh]. If the Palais–Smale condition fails, the corresponding functions concentrate at a finite number of points. Such sequences have been carefully analyzed, see e.g. Brezis and Nirenberg [B-N], Brezis and Coron [Br-Co] and Struwe [St1,2]. In order to bypass the lack of compactness and apply variational techniques, such as Morse theory, Bahri and Coron [Ba-Co] have been led to a new tool: the critical points at infinity.

(3) The *complex Monge–Ampère equation* has the same form as the real Monge–Ampère equation:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i\, \partial z_{\bar{j}}}\right) = e^{F(\cdot,\, u)} \det(g_{i\bar{j}}) \qquad \text{on } M,$$

where M is a compact complex manifold with a Kähler metric $\sum g_{i\bar{j}}\,dz_i \otimes dz_{\bar{j}}$, u is a real valued unknown and F is a given function on $M \times \mathbb{R}$. This equation arises in the study of Calabi's conjecture which asserts that every form representing the first Chern class $C_1(M)$ is the Ricci form of some Kähler metric on M. Calabi's conjecture was established in the mid-1970's by S. T. Yau in case the first Chern class is vanishing and by Aubin and Yau, independently, in case the first Chern class is negative; see [Au] and [Ya]. The method of Yau used ideas developed earlier by Calabi and Pogorelov for the real Monge–Ampère equation. On the other hand, Yau's approach was a great stimulus for the completion of the study of the real Monge–Ampère equation (see Section 23). A complex Monge–Ampère equation also occurs in the work of Fefferman [Fe] on the Bergman kernel for several complex variables. See also the work of Hamilton [Ham2] on Ricci flows.

(4) *The Yang–Mills equations* correspond to the Euler–Lagrange equation of the Yang–Mills functional

$$YM(A) = \int_M |F_A|^2$$

where $F_A = dA + A \wedge A$ is the curvature of a connection A. From the point of view of calculus of variations this is again a borderline case for compactness when $\dim M = 4$ (because of the Sobolev imbedding $H^1 \subset L^4$). This equation is of importance in the description of elementary particles as proposed by Yang and Mills in 1954. It has also found a remarkable application in Donaldson's study of four dimensional manifolds.

The Yang–Mills equations and, more generally, gauge theory have been extensively investigated since the mid-1970's by a number of people including M. Atiyah, S. K. Donaldson, C. Taubes, K. Uhlbenbeck, E. Witten, N. Seiberg and many others.

25. COMPUTATION OF SOLUTIONS OF PDE'S: NUMERICAL ANALYSIS AND COMPUTATIONAL SCIENCE

One of the most important and striking phenomena of the applications of PDE's in the physical sciences and engineering since the second world war has been the impact of high speed digital computation. Despite the strikingly optimistic predictions of some of the pioneers and prophets in the field, including J. van Neumann, this has not turned out to be a panacea for all the problems of the field. It has however drastically changed the structure of practice in applied mathematics and has given rise to new problems and new perspectives. In some cases, e.g. meteorology (an area in which van Neumann saw the greatest theoretical consequences for the digital computers) the worldwide practice of meteorological prediction has been absorbed into the dual task of gathering atmospheric data over a planetary basis and analyzing it using the most elaborate supercomputers available. On the other hand, it has become increasingly clear, on the basis of intrinsically chaotic structures of the situation (as in the theory of chaos, first applied to this domain by E. Lorenz) that there are inherent limitations, in principle, to long term predictions which cannot be overcome simply by massive computing power.

On a practical level, almost all PDE's are studied by computational means. Such studies take one of two forms, which are somewhat discordant in practice. The first and narrower form is that of classical *numerical analysis*, a branch of analytical applied mathematics, which obtains results with error bounds on the basis of relatively rigorous arguments. It uses finite dimensional function spaces and relies on solving the approximate problem in the finite dimensional context. Another mode of practice, which is used on a broader scale, is the approach which is often described as *computational science*. In this approach one sets up simplified computational models for the given equations and one computes the solution in the simplified situations without attempting to obtain a strict control of the mathematical validity of this process. The justification is in terms of the phenomenology of the results, although this often gives rise to ambiguity and to controversies about the validity of the computational process, particularly in situations which are difficult to analyze from a theoretical standpoint, e.g. turbulence.

In summary, the situation has seen the development of an enormously powerful tool to obtain concrete results on PDE's arising in a variety of applied contexts, but the tool itself in its application has created very difficult problems to be resolved in the future. A compensating feature of the new situation is the use of computations as an *experimental instrument to generate conjectures* for analytic arguments and to study the numerical simulations as a source of suggestions for rigorous treatment. Thus, as in all fields of science, the triad of methodologies, theoretical, experimental and computational, must be integrated to make possible an adequate attack upon the most difficult and most fundamental problems.

REFERENCES

For a broad presentation of the current state of the theory of PDE's we refer to the books of (1) Hörmander [Hor4] and Taylor [Ta1,2] for linear PDE's, and (2) Morrey [Mor2], Gilbarg and Trudinger [G-T] and Giaquinta [Gia1,2] for nonlinear elliptic PDE's.

[Ad] R. A. Adams, "Sobolev Spaces," Academic Press, San Diego, 1975.

[A-H] D. R. Adams and L. I. Hedberg, "Function Spaces and Potential Theory," Springer-Verlag, Berlin, 1996.

[A-D-N] S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II, *Comm. Pure Appl. Math.* **12** (1959), 623–727; **17** (1964), 35–92.

[A-R] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.

[Ant] S. Antman, "Nonlinear Problems of Elasticity," Springer-Verlag, Berlin, 1995.

[Arn] V. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier* **16** (1966), 319–361.

[A-W] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.* **30** (1978), 33–58.

[A-S] M. F. Atiyah and I. M. Singer, The index of elliptic operators I, III, *Ann. Math.* **87** (1986), 484–530; 546–604.

[Au] T. Aubin, "Nonlinear Analysis on Manifolds, Monge–Ampère Equations," Springer-Verlag, Berlin, 1982.

[Ba-Co] A. Bahri and J. M. Coron, Une théorie des points critiques à l'infini pour l'équation de Yamabe et le problème de Kazdan–Warner, *C.R. Acad. Sci. Paris* **300** (1985), 513–516.

[Ba] C. Baiocchi, Su un problema a frontiere libera connesso a questioni di idraulica, *Ann. Math. Pura Appl.* **92** (1972), 107–127.

[Ba-Ca] C. Baiocchi and A. Capelo, "Disequazioni variazionali e quasivariazionali," Vol. 1, 2, Pitagora, 1978 [Italian]; Engl. transl. Wiley, New York, 1984.

[Bal] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rat. Mech. Anal.* **63** (1977), 337–403.

[Bar] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff, Groningen, 1976.

[B-F] R. Beals and C. Fefferman, On local solvability of linear partial differential equations, *Ann. Math.* **97** (1973), 482–498.

[B-H-W] J. Bemelmans, S. Hildebrandt, and W. von Wahl, Partielle differentialgleichungen und variationsrechnung, *in* "Ein Jahrhundert Mathematik, 1890–1990," Vieweg, 1990.

[B-B-M] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equation for long waves and nonlinear dispersive systems, *Phil. Trans. R. Soc. London, Ser. A* **272** (1972), 47–78.

[Be-N] H. Berestycki and L. Nirenberg, Travelling fronts in cyclinders, *Ann. IHP Anal. Nonlin.* **9** (1992), 497–572.

[Be1] S. N. Bernstein, Sur la nature analytique des solutions de certaines équations aux dérivées partielles, *Math. Ann.* **59** (1904), 20–76.

[Be2] S. N. Bernstein, Sur la généralisation du problème de Dirichlet, I, II, *Math. Ann.* **62** (1906), 253–272; **69** (1910), 82–136.

[B-B-H] F. Bethuel, H. Brezis, and F. Hélein, "Ginzburg–Landau Vortices," Birkhäuser, Basel, 1994.

[B-D-G] E. Bombieri, E. DeGiorgi, and E. Giusti, Minimal cones and the Bernstein problem, *Inv. Math.* **7** (1969), 243–268.

[Bon] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux derivées partielles nonlinéaires, *Ann. Sc. Ec. Norm. Sup.* **14** (1981), 209–246.

[B-B] J. P. Bourguignon and H. Brezis, Remarks on the Euler equations, *J. Funct. Anal.* **15** (1974), 341–362.

[Bre1] H. Brezis, Equations et inéquations nonlinéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier* **18** (1968), 115–175.

[Bre2] H. Brezis, "Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam, 1973.

[Bre3] H. Brezis, Some variational problems with lack of compactness, *in* "Nonlinear Functional Analysis and Its Applications" (F. Browder, Ed.), Proc. Symp. Pure Math., Vol. 45, Am. Math. Soc., Providence, 1986.

[B-C] H. Brezis and T. Cazenave, "Semilinear Evolution Equations and Applications in Mechanics and Physics," Pitman Lecture Notes, Addison–Wesley, Reading, MA, to appear.

[Br-Co] H. Brezis and J. M. Coron, Convergence of solutions of H-systems or how to blow bubbles, *Arch. Rat. Mech. Anal.* **89** (1985), 21–56.

[B-C-L] H. Brezis, J. M. Coron, and E. Lieb, Harmonic maps with defects, *Comm. Math. Phys.* **107** (1986), 649–705.

[B-N] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[B-S] H. Brezis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, *Bull. Soc. Math. Fr.* **96** (1968), 153–180.

[Bro1] F. Browder, The Dirichlet problem for linear elliptic equations of arbitrary even order with variable coefficients, *Proc. Nat. Acad. Sc. USA* **38** (1952), 230–235; On the eigenfunctions and eigenvalues of the general linear elliptic differential operator, *Proc. Nat. Acad. Sc. USA* **39** (1953), 433–439.

[Bro2] F. Browder, The eigenfunction expansion theorem for the general self-adjoint singular elliptic partial differential operator, *Proc. Nat. Acad. Sci. USA* **40** (1954), 454–459.

[Bro3] F. Browder, Estimates and existence theorems for elliptic boundary value problems, *Proc. Nat. Acad. Sc. USA* **45** (1959), 365–372.

[Bro4] F. Browder, On the spectral theory of elliptic differential operators, I, *Math. Ann.* **142** (1961), 22–130; II, *Math. Ann.* **143** (1962), 81–226.

[Bro5] F. Browder, Nonlinear elliptic boundary value problems, *Bull. Amer. Math. Soc.* **69** (1963), 862–874.

[Bro6] F. Browder, "Nonlinear Operators and Nonlinear Equations of Evolution," Proc. Symp. Pure Math., Vol. 18, Part 2, Amer. Math. Soc., Providence, RI, 1976.

[Bro7] F. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.* **9** (1983), 1–39.

[B-P] F. Browder and W. Petryshyn, Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces, *J. Funct. Anal.* **3** (1969), 217–245.

[Ca1] R. Cacciopoli, Sulle equazioni ellittiche a derivate parziali con n variabili independenti, *Rend. Accad. Lincei* **19** (1934), 83–89.

[Ca2] R. Cacciopoli, Sulle corrispondenze funzionali inverse diramate: teoria generale e applicazioni ad alcune equazioni funzionali nonlineari e al problema di Plateau, *Rend. Acc. Naz. Lincei, s. VI* **24** (1936), 258–263; 416–421.

[C-C] L. Caffarelli and X. Cabré, Fully nonlinear elliptic equations, *Amer. Math. Soc. Colloq. Ser.* **43** (1995).

[C-K-N] L. Caffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, *Comm. Pure Appl. Math.* **35** (1982), 771–831.

[C-N-S] L. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, I: Monge–Ampère equation, *Comm. Pure Appl. Math.* **37** (1984), 369–402.

[Cal1] A. P. Calderon, Uniqueness in the Cauchy problem for partial differential equations, *Amer. J. Math.* **80** (1958), 16–36.

[Cal2] A. P. Calderon, Existence and uniqueness theorems for systems of partial differential equations, *in* "Fluid Dynamics and Applied Mathematics," Proc. Symp. Univ. of Maryland, 1961.

[C-Z] A. P. Calderon and A. Zygmund, On the existence of certain singular integrals, *Acta Math.* **88** (1952), 85–139.

[Car] T. Carleman, Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes, *in* “Förhandlingar Skandinaviska Matematikerkongressen, Stockholm, 1934,” pp. 34–44.

[Ch] K. C. Chang, “Infinite Dimensional Morse Theory and Multiple Solution Problems,” Birkhäuser, Basel, 1993.

[C-K] D. Christodoulou and S. Klainerman, “The Global Nonlinear Stability of Minkowski Space,” Princeton Univ. Press, Princeton, NJ, 1993.

[Co] R. Courant, “Dirichlet’s Principle, Conformal Mapping and Minimal Surfaces,” Interscience–Wiley, New York, 1970.

[C-F] R. Courant and K. Friedrichs, “Supersonic Flow and Shock Waves,” Interscience–Wiley, New York, 1948.

[C-H] R. Courant and D. Hilbert, “Methoden der Matematischen Physik,” Springer-Verlag, Berlin, 1937; English version, “Methods of Mathematical Physics, I, II,” Interscience–Wiley, New York, 1962.

[C-L] M. Crandall and T. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, *Amer. J. Math.* **93** (1971), 265–298.

[Cr-L] M. Crandall and P. L. Lions, Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), 1–42.

[Dac] B. Dacorogna, “Weak Continuity and Weak Lower Semi-continuity of Nonlinear Functionals,” Springer-Verlag, Berlin, 1987.

[Da1] E. B. Davies, “One Parameter Semigroups,” Academic Press, San Diego, 1980.

[Da2] E. B. Davies, L^p spectral theory of higher order elliptic differential operators, *Bull. London Math. Soc.* **29** (1997), 513–546.

[Dg1] E. DeGiorgi, Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari, *Mem. Accad Sc. Torino, C. Sc. Fis. Mat. Natur.* **3** (1957), 25–43.

[Dg2] E. DeGiorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellitico, *Bull. UMI* **4** (1968), 135–137.

[D-H-K-W] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, “Minimal Surfaces,” Vol. I, II, Springer-Verlag, Berlin, 1992.

[Di] J. Dieudonné, “History of Functional Analysis,” North-Holland, Amsterdam, 1981.

[DiP] R. DiPerna, Compensated compactness and general systems of conservation laws, *Trans. Amer. Math. Soc.* **292** (1985), 383–419.

[D-L] R. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability, *Ann. Math.* **130** (1989), 321–366.

[D-M] R. DiPerna and A. Majda, Reduced Hausdorff dimension and concentration cancellation for two-dimensional incompressible flow, *J. Amer. Math. Soc.* **1** (1988), 59–95.

[Do] J. Douglas, Solution of the problem of Plateau, *Trans. Amer. Math. Soc.* **33** (1931), 263–321.

[Du] J. J. Duistermaat, “Fourier Integral Operators,” Birkhäuser, Basel, 1996.

[D-G] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975), 39–79.

[E-M] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.* **92** (1970), 102–163.

[E-S] Yu. V. Egorov and M. A. Shubin, “Partial Differential Equations, IV,” Springer-Verlag, Berlin, 1993.

[Eh] L. Ehrenpreis, Solutions of some problems of division I, *Amer. J. Math.* **76** (1954), 883–903.

[Ek] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353.

[Ek-Te] I. Ekeland and R. Temam, "Convex Analysis and Variational Problems," North-Holland, Amsterdam, 1976.

[E-T] K. D. Elworthy and A. J. Tromba, Differential structures and Fredholm maps on Banach manifolds, *in* "Global Analysis," Proc. Symp. Pure Math., Vol. 15, Am. Math. Soc., Providence, RI, 1969.

[Ev1] L. C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, *Comm. Pure Appl. Math.* **25** (1982), 333–363.

[Ev2] L. C. Evans, Weak convergence methods for nonlinear partial differential equations, *CBMS Regional Conf. Ser. in Math.* **74** (1988).

[F-T] L. D. Fadeev and L. A. Takhtajan, "Hamiltonian Methods in the Theory of Solutions," Nauka, Moscow, 1986 [Russian]; Engl. transl., Springer-Verlag, Berlin, 1987.

[Fe] C. Fefferman, Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, *Ann. Math.* **103** (1976), 395–416.

[Fis] E. Fischer, Über quadratische Formen met reellen Koeffizienten, *Monatsh. Math. Phys.* **16** (1905), 234–249.

[F-Z] A. S. Fokas and V. E. Zakharov (Eds), "Important Developments in Soliton Theory," Springer-Verlag, Berlin, 1993.

[Fr] A. Friedman, "Variational Principles and Free-Boundary Problems," Wiley, New York, 1982.

[Fd] K. Friedrichs, On the differentiability of the solutions of linear elliptic differential equations, *Comm. Pure Appl. Math.* **6** (1953), 299–326.

[Fu] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sc. Univ. Tokyo Sect. I A* **13** (1966), 109–124.

[Ga1] E. Gagliardo, Proprietà di alcune classi di funzioni di più variabili, *Ric. di Mat.* **7** (1958), 102–137.

[Ga2] E. Gagliardo, Ulteriori proprietà di alcune classi di funzioni in più variabili, *Ric. di Mat.* **8** (1959), 24–51.

[Gå1] L. Gårding, Dirichlet's problem for linear elliptic partial differential equations, *Math. Scand.* **1** (1953), 55–72.

[Gå2] L. Gårding, On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators, *Math. Scand.* **1** (1953), 237–255.

[Gå3] L. Gårding, Eigenfunction expansions connected with elliptic differential operators, *Proc. 2nd Scand. Math. Congr. Lund* (1954), 44–55.

[Gå4] L. Gårding, Solution directe du problème de Cauchy pour les équations hyperboliques, *Coll. Int. CNRS, Nancy* (1956), 71–90.

[G-G-K-M] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* **19** (1967), 1095–1097.

[Ge] I. M. Gelfand, Some problems in the theory of quasilinear equations, *Usp. Mat. Nauk.* **14** (1959), 87–158, Engl. transl., *Amer. Math. Soc. Transl. Ser. 2* (1963), 295–381.

[G-L] I. M. Gelfand and B. Levitan, On the determination of a differential operator from its spectral function, *Amer. Math. Soc. Transl. (2)* **1** (1955), 253–304.

[G-S] I. M. Gelfand and G. Shilov, "Generalized Functions," 5 volumes, Russian edition 1958; Engl. transl. Academic Press, San Diego, 1967.

[Gia1] M. Giaquinta, "Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems," Princeton Univ. Press, Princeton, NJ, 1983.

[Gia2] M. Giaquinta, "Introduction to Regularity Theory for Nonlinear Elliptic Systems," Birkhäuser, Basel, 1993.

[G-N-N] B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **6** (1981), 883–901.

[G-T] D. Gilbarg and N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer-Verlag, Berlin, 1983.

[Giu] E. Giusti, "Minimal Surfaces and Functions of Bounded Variation," Birkhäuser, Basel, 1984.

[G-M] E. Giusti and M. Miranda, Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regulare del calcolo delle variazioni, *Bull. UMI* **2** (1968), 1–8.

[Gl] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 95–105.

[Gl-Ja] J. Glimm and A. Jaffe, "Quantum Physics: A Functional Integral Point of View," 2nd ed., Springer-Verlag, Berlin, 1987.

[Gl-La] J. Glimm and P. Lax, "Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws," Amer. Math. Soc. Memoir No. 101, Am. Math. Soc., Providence, 1970.

[Go] J. A. Goldstein, "Semigroups of Linear Operators and Applications," Oxford Univ. Press, London, 1985.

[G-W-W] C. Gordon, L. Webb, and S. Wolpert, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* **27** (1992), 134–138.

[Ha] J. Hadamard, "Lectures on Cauchy's Problem in Linear Partial Differential Equations," Yale Univ. Press, New Haven, CI, 1923, Reprinted, Dover, 1952.

[Ham1] R. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* **7** (1982), 65–222.

[Ham2] R. Hamilton, "The Formation of Singularities in the Ricci Flow," Surveys in Differential Geometry, Vol. II, pp. 7–136, International Press, 1995.

[Hi1] D. Hilbert, Über das Dirichletsche Prinzip, *Jahresbericht Deut Math.-Ver.* **VIII** (1900), 184–188. Also in *J. Reine Angew. Math.* **129** (1905), 63–67 and *Math. Ann.* **59** (1904), 161–184.

[Hi2] D. Hilbert, "Grundzüge einer allgemeinen Theory der linearen Integralgleichungen," Leipzig, 1912.

[H-P] E. Hille and R. S. Phillips, Functional analysis and semi-groups, *Amer. Math. Soc. Coll. Publ.* **31** (1957).

[Hol] O. Hölder, "Beiträge zur Potentialtheorie," Stuttgart, 1882.

[Hop1] E. Hopf, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitz. Preuss. Akad. Wiss.* **19** (1927), 147–152.

[Hop2] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.* **4** (1950/51), 213–231.

[Hop3] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.* **3** (1950), 201–230.

[Hor1] L. Hörmander, On the theory of general partial differential equations, *Acta Math.* **94** (1955), 161–248.

[Hor2] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121** (1968), 193–218.

[Hor3] L. Hörmander, Fourier integral operators I, *Acta Math.* **127** (1971), 79–183.

[Hor4] L. Hörmander, "The Analysis of Linear Partial Differential Operators," Vol. 1-4, Springer-Verlag, Berlin, 1983–85.

[I] V. Ivrii, On the second term in the spectral asymptotics for the Laplace-Beltrami operator on a manifold with boundary, *Funk. Anal. i Prilozen* **14** (1980), 25–34 [Russian]; Engl. trans., *Funct. Anal. Appl.* **14** (1980), 98–106.

[J1] F. John, "Plane Waves and Spherical Means Applied to Partial Differential Equations," Interscience, New York, 1955. Reprinted, Springer-Verlag, Berlin, 1981.

[J2] F. John, "Partial Differential Equations," Springer-Verlag, Berlin, 1971.

[Ka] M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* **73** (1966), 1–23.

[Ka-Sc] M. Kashiwara and P. Schapira, "Sheaves on Manifolds," Springer-Verlag, Berlin, 1990.

[Ka1] T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin, 1966.

[Ka2] T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation, *Archive Rat. Mech. Anal.* **25** (1967), 188–200.

[Ke] O. D. Kellogg, "Foundations of Potential Theory," Springer-Verlag, Berlin, 1929. Reprinted Dover, 1953.

[Ken] C. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, *CBMS/AMS Ser.* **83** (1994).

[K-S] D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inequalities and Their Applications," Academic Press, San Diego, 1980.

[Kli] M. Kline, "Mathematical Thought from Ancient to Modern Times," Vol. 1, 2, 3, Oxford Univ. Press, London, 1972.

[Koh] J. J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds, I, II, *Ann. Math.* **78** (1963), 112–148; **79** (1964), 450–472.

[K-N] J. J. Kohn and L. Nirenberg, On the algebra of pseudo-differential operators, *Comm. Pure Appl. Math.* **18** (1965), 219–305.

[K-P-P] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. Moscou*, *Sec. A* **1** (1937), 1–26.

[Kor] A. Korn, Sur les équations de l'elasticité, *Ann. Sc. Ec. Norm. Sup.* **24** (1907).

[Kra] M. Krasnoselskii, "Topological Methods in the Theory of Nonlinear Integral Equations," Gostehizdat, 1956 [Russian]; Engl. transl., Macmillan, New York, 1964.

[Kry1] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations, *Izv. Akad. Nauk SSSR* **46** (1982), 487–523; **47** (1983), 75–108 [Russian]; Engl. transl., *Math. USSR Izv.* **20** (1983), 459–492; **22** (1984), 67–97.

[Kry2] N. V. Krylov, On degenerate nonlinear elliptic equations, *Mat. Sb.* **120** (1983), 311–330 [Russian]; Engl. transl., *Math. USSR Sb.* **48** (1984), 307–326.

[Kry3] N. V. Krylov, "Nonlinear Elliptic and Parabolic Equations of Second Order," Nauka, Moscow, 1985 [Russian]; Engl. transl., Reidel, Dordrecht, 1987.

[Kr-Sa] N. V. Krylov and M. V. Safonov, Certain properties of solutions of parabolic equations with measurable coefficients, *Izv. Akad. Nauk. SSSR* **44** (1980), 161–175 [Russian]; Engl. transl., *Math. USSR Izv.* **16** (1981), 151–164.

[L1] O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," Gos. Izdat. Fiz-Mat., Moscow, 1961 [Russian]; Engl. transl., Gordon and Breach, 2nd ed., 1969; Engl. transl. *Math. USSR*, *Izv.* **20** (1983), 459–492.

[L2] O. A. Ladyzhenskaya, "Attractors for Semigroups and Evolution Equations," Cambridge Univ. Press, Cambridge, UK, 1991.

[L-U] O. A. Ladyzhenskaya and N. N. Uraltseva, "Linear and Quasilinear Elliptic Equations," Nauka, Moscow, 1964 [Russian]; English transl., Academic Press, San Diego, 1968.

[Lan-M] R. Landes and V. Mustonen, On the pseudomonotone operators and nonlinear noncoercive problems on unbounded domains, *Math. Ann.* **248** (1980), 241–246.

[La1] P. Lax, On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations, *Comm. Pure Appl. Math.* **8** (1955), 615–633.

[La2] P. Lax, Asymptotic solutions of oscillatory initial value problems, *Duke Math. J.* **24** (1957), 627–646.

[La3] P. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[La4] P. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467–490.

[L-M] P. Lax and A. N. Milgram, Parabolic equations, *in* "Contributions to the Theory of Partial Differential Equations," Ann. Math. Studies, Vol. 33, pp. 167–190, Princeton Univ. Press, Princeton, NJ, 1954.

[L-P] P. Lax and R. Phillips, "Scattering Theory," Academic Press, San Diego, 1967.

[Le1] J. Leray, Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appli. Ser. IX* **12** (1933), 1–82.

[Le2] J. Leray, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, *J. Math. Pures Appl. Ser. IX* **13** (1934), 331–418.

[Le3] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193–248.

[Le4] J. Leray, "Hyperbolic Equations with Variable Coefficients," Lecture Notes, Inst. Adv. Study, Princeton, 1952.

[L-L] J. Leray and J. L. Lions, Quelques résultats de Visik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty–Browder, *Bull. Soc. Math. Fr.* **93** (1965), 97–107.

[L-S] J. Leray and J. P. Schauder, Topologie et équations fonctionnelles, *Ann. Sci. Ec. Norm. Sup.* **51** (1934), 43–78.

[LB] B. Levi, Sul principio di Dirichlet, *Rend. Circ. Mat. Palermo* **22** (1906), 293–359.

[Le] E. E. Levi, Sulle equazioni lineare totalmente ellitiche, *Rend. Circ. Mat. Palermo* **24** (1907), 275–317.

[Lew] H. Lewy, An example of a smooth linear partial differential equation without solution, *Ann. Math.* **66** (1957), 155–158.

[Le-St] H. Lewy and G. Stampacchia, On the regularity of the solution of a variational inequality, *Comm. Pure Appl. Math.* **22** (1969), 153–188.

[Li1] L. Lichtenstein, Über den analytischen Charakter der Lösungen Zweidimensionaler Variationsproblem, *Bull. Acad. Sci. Cracovie Cl. Sci. Mat. Nat.* **A** (1912), 915–941.

[Li2] L. Lichtenstein, Neuere Entwicklung der Potentialtheorie. Konforme Abbildung, *Encyklop. der Math. Wiss. II C* **3** (1918), 177–377.

[Li3] L. Lichtenstein, Neuere Entwicklung der Theorie Partieller Differentialgleichungen Zweiter Ordnung von Elliptischen Typus, *Encyklop. der Math. Wiss. II C* **12** (1924), 1277–1334.

[Li4] L. Lichtenstein, Über einige Existenzprobleme der Hydrodynamik homogener unzusammendrückbarer, reibungloser Flüssigkeiten und die Helmholtzschen Wirbelsatze, *Math. Z.* **23** (1925), 89–154; 309–316; **26** (1927), 196–323; **28** (1928), 387–415.

[Lio1] J. L. Lions, Problèmes aux limites en théorie des distributions, *Acta Math.* **94** (1955), 13–153.

[Lio2] J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites nonlinéaires," Dunod–Gauthier Villars, Paris, 1969.

[Li-Ma] J. L. Lions and E. Magenes, "Problèmes aux limites non homogènes et applications," Vol. 1–2, Dunod, Paris, 1968; Engl. transl., Springer-Verlag, Berlin, 1972.

[Lj-Sc] L. Ljusternik and L. Schnirelman, "Méthodes topologiques dans les problèmes variationnels," Hermann, Paris, 1934.

[Lo] Ya. B. Lopatinski, On a method of reducing boundary problems for a system of differential equations of elliptic type to regular integral equations, *Ukrain. Math. J.* **5** (1953), 123–151 [Russian]; Engl. transl., *Amer. Math. Transl. (2)* **89** (1970), 149–183.

[M-S] H. McKean and I. Singer, Curvature and the eigenvalues of the Laplacian, *J. Diff. Geom.* **1** (1967), 43–69.

[Mal] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et équations de convolution, *Ann. Inst. Fourier* **6** (1955–56), 271–355.

[Mas] V. P. Maslov, "Perturbation Theory and Asymptotic Methods," Mos. Gozd. Univ., 1965 [Russian]; French transl., Dunod, Paris, 1972.

[Mau] F. Mautner, On eigenfunction expansions, *Proc. Nat. Acad. Sci.* **39** (1952), 49–53.

[M-W] J. Mawhin and M. Willem, "Critical Point Theory and Hamiltonian Systems," Springer-Verlag, Berlin, 1989.

[Maz1] V. G. Mazya, Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients, *Funk. Anal. i Prilozen.* **2** (1968), 53–57. [In Russian; Engl. transl., *Funct. Anal. Appl.* **2** (1968), 230–234.]

[Maz2] V. G. Mazya, "Sobolev Spaces," Springer-Verlag, Berlin, 1985.

[Me] Y. Meyer, "Ondelettes et Opérateurs," Vol. 3, Hermann, Paris, 1990.

[M-S] N. Meyers and J. Serrin, $H=W$, *Proc. Nat. Acad. Sci. USA* **51** (1964), 1055–1056.

[Mik] S. G. Mikhlin, Singular integral equations, *Uspekhi Mat. Nauk* **3** (1948), 29–112 [Russian]; Engl. transl., *Amer. Math. Soc. Trans. Ser. 1* **10** (1962), 84–198.

[M-P] S. Minakshisundaram and A. Pleijel, Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds, *Can. J. Math.* **1** (1949), 242–256.

[Min] G. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.* **29** (1962), 341–346.

[Mo-St] C. Morawetz and W. Strauss, Decay and scattering of solutions of nonlinear relativistic wave equations, *Comm. Pure Appl. Math.* **25** (1972), 1–31; **26** (1973), 47–54.

[Mor1] C. B. Morrey, Jr., On the solutions of quasilinear elliptic partial differential equations, *Trans. Amer Math. Soc.* **43** (1938), 126–166.

[Mor2] C. B. Morrey, Jr., "Multiple Integral Problems in the Calculus of Variations and Related Topics," Univ. of California Publ. in Math., new ser. I, pp. 1–130, (1943).

[Mor3] C. B. Morrey, Jr., Quasiconvexity and the lower semicontinuity of multiple integrals, *Pacific J. Math.* **2** (1952), 25–53.

[Mor4] C. B. Morrey, Jr., "Multiple Integrals in the Calculus of Variations," Springer-Verlag, Berlin, 1966.

[Mrs1] M. Morse, Relations between the critical points of a real function of n independent variables, *Trans. Amer. Math. Soc.* **27** (1925), 345–396.

[Mrs2] M. Morse, The calculus of variations in the large, *Amer. Math. Soc. Colloq. Publ.* **18** (1934).

[Mos1] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.* **13** (1960), 457–468.

[Mos2] J. Moser, A rapidly convergent iteration method and nonlinear partial differential equations, I, II, *Ann. Sc. Norm. Sup. Pisa* **20** (1966), 226–315; 449–535.

[Na1] J. Nash, The embedding problem for Riemann manifolds, *Ann. Math.* **63** (1956), 20–63.

[Na2] J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.* **80** (1958), 931–954.

[Nec] J. Necas, "Introduction to the Theory of Nonlinear Elliptic Equations," Teubner and Wiley, New York, 1983.

[New] A. C. Newell, "Solitons in Mathematics and Physics," CBMS-NSF, Reg. Conf. in Applied Math. SIAM, 1985.

[Ni1] L. Nirenberg, On nonlinear elliptic partial differential equations and Hölder continuity, *Comm. Pure Appl. Math.* **6** (1953), 103–156.

[Ni2] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Pisa* **13** (1959), 115–162.

[Ni3] L. Nirenberg, Partial differential equations in the first half of the century, *in* "Development of Mathematics 1900-1950" (J. P. Pier, Ed.), Birkhäuser, Basel, 1994.

[N-T] L. Nirenberg and F. Treves, On local solvability of linear partial differential equations, I, II, *Comm. Pure Appl. Math.* **23** (1970), 1–38; 459–509; **24** (1971), 279–288.

[Nit] J. C. C. Nitsche, "Lectures on Minimal Surfaces," Cambridge Univ. Press, Cambridge, UK, 1989.

[Ol] O. Oleinik, Discontinuous solutions of nonlinear differential equations, *Uspekhi Mat. Nauk* (*N.S.*) **12** (1957), 3–73; Engl. transl., *Amer. Math. Soc. Transl. Ser. 2* **26**, 95–172.

[P-S] R. Palais and S. Smale, A generalized Morse theory, *Bull. Amer. Math. Soc.* **70** (1964), 165–171.

[Pa] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, Berlin, 1983.

[Per] O. Perron, Eine neue Behandlung der Randwertaufgabve für $\Delta u = 0$, *Math. Z.* **18** (1923), 42–54.

[Pet] I. G. Petrovskii, On the Cauchy problem for a system of linear partial differential equations in the realm of nonanalytic functions, *Bull. Moscow State Univ. Sec. A* **1** (1938), 1–72. [Russian]

[Poh] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Mat. Dokl.* **6** (1965), 1408–1411.

[Po1] H. Poincaré, Sur les équations aux dérivées partielles de la physique mathématique, *Amer. J. Math.* **12** (1890), 211–294.

[Po2] H. Poincaré, Sur les équations de la physique mathématique, *Rend. Circ. Mat. Palermo* **8** (1894), 57–155.

[Po3] H. Poincaré, Les fonctions fuchsiennes et l'équation $\Delta u = e^u$, *J. Math. Pures Appl. 5e ser.* **4** (1898), 137–230.

[Po4] H. Poincaré, "Théorie du Potential Newtonien," Carré et Naud, 1899; Reprinted, J. Gabag, 1990.

[P-W] M. H. Protter and H. F. Weinberger, "Maximum Principles in Differential Equations," Prentice Hall, New York, 1967.

[Ra1] P. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.

[Ra2] P. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," CBMS Reg. Conf. Ser. Math., Vol. 65, Amer. Math. Soc., Providence, 1986.

[R-S] M. Reed and B. Simon, "Methods of modern mathematical physics," Vol. 1–4, Academic Press, San Diego, 1972-1979.

[RiF] F. Riesz, Über lineare Funktionalgleichungen, *Acta Math.* **41** (1917), 71–98.

[RiM] M. Riesz, L'intégrale de Riemann–Liouville et le problème de Cauchy, *Acta Math.* **81** (1949), 1–223.

[Sa-Uh] J. Sacks and K. Uhlenbeck, The existence of minimal 2-spheres, *Ann. Math.* **113** (1981), 1–24.

[S-V] Yu. Safarov and D. Vassiliev, "The Asymptotic Distribution of Eigenvalues of Partial Differential Operators," Transl. Math. Monographs, Vol. 155, Amer. Math. Soc., Providence, 1996.

[Sca1] J. P. Schauder, Der Fixpunksatz in Funkionalräume, *Studia Math.* **2** (1930), 171–180.

[Sca2] J. P. Schauder, Über lineare elliptische Differentialgleichungen sweiter Ordnung, *Math. Z.* **38** (1934), 257–282.

[Sce] M. Schechter, Various types of boundary conditions for elliptic equations, *Comm. Pure Appl. Math.* **13** (1960), 407–425.

[Sco] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Diff. Geom.* **20** (1964), 479–495.

[S-U] R. Schoen and K. Uhlenbeck, A regularity theory for harmonic maps, *J. Diff. Geom.* **17** (1982), 307–335.

[Scw] L. Schwartz, "Théorie des distributions I, II," Hermann, Paris, 1950, 1951.

[Seg] I. Segal, Nonlinear semi-groups, *Ann. Math.* **78** (1963), 339–364.

[Ser1] J. Serrin, The problem of Dirchlet for quasilinear elliptic differential equations with many independent variables, *Phil. Trans. Royal Soc. London, Ser. A* **264** (1969), 413–496.

[Ser2] J. Serrin, A symmetry problem in potential theory, *Arch. Rat. Mech. Anal.* **43** (1971), 304–318.

[Sh] Z. Shapiro, On general boundary problems for equations of elliptic type, *Izv. Akad. Nauk SSSR* **17** (1953), 539–562. [Russian]

[Sk] I. V. Skrypnik, "Methods of Analysis of Nonlinear Elliptic Boundary Value Problems," Transl. Math. Monographs, Vol. 139, Amer. Math. Soc., Providence, 1991; Russian edition, 1990.

[Sma] S. Smale, An infinite dimensional version of Sard's theorem, *Amer. J. Math.* **87** (1965), 861–866.

[Smo] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, Berlin, 1983.

[So1] S. L. Sobolev, On a theorem of functional analysis, *Mat. Sb.* **46** (1938), 471–497 [Russian]; Engl. transl., *Amer. Math. Soc. Trans. (2)* **34**, 1963, 39–68.

[So2] S. L. Sobolev, "Applications of Functional Analysis in Mathematical Physics," Izdat. Leningrad Gos. Univ., 1950 [Russian]; Engl. transl., Trans. Math. Monographs, Vol. 7, Amer. Math. Soc., Providence, 1963.

[Sta] G. Stampacchia, "Equations elliptiques du second ordre à coéfficients discontinus," Presses Univ. Montreal, 1966.

[Ste] E. Stein, "Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals," Princeton Univ. Press, Princeton, NJ, 1993.

[Str1] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984), 511–517.

[Str2] M. Struwe, "Variational Methods," Springer-Verlag, Berlin, 1990.

[Sy-Uh] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. Math.* **125** (1987), 153–169.

[Ta1] M. Taylor, "Pseudodifferential Operators," Princeton Univ. Press, Princeton, NJ, 1981.

[Ta2] M. Taylor, "Partial Differential Equations, I, II, III," Springer-Verlag, Berlin, 1996.

[Te1] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.* **20** (1975), 32–43.

[Te2] R. Temam, "Navier–Stokes equations, Theory and Numerical Analysis," rev. ed., North-Holland, Amsterdam, 1979.

[Te3] R. Temam, "Infinite Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, Berlin, 1988.

[Tre] F. Treves, "Introduction to Pseudodifferential and Fourier Integral Operators," Vol. I, II, Plenum, New York, 1980.

[Tru] C. Truesdell, A program toward rediscovering the rational mechanics of the age of reason, *Arch. Hist. Exact Sc.* **1** (1960), 3–36.

[Ve] L. Véron, "Singularities of Solutions of Second Order Quasilinear Equations," Longman, Harlow/New York, 1996.

[Vi1] M. I. Visik, On strongly elliptic systems of differential equations, *Mat. Sb.* **29** (1951), 615–676.

[Vi2] M. I. Visik, Quasi-linear strongly elliptic systems of differential equations in divergence form, *Trudi Mosk. Math. Ob.* **12** (1963), 125–184 [Russian]; Engl. transl., *Transl. Moscow Math. Soc.* **12** (1963).

[Wei] F. B. Weissler, Single point blow-up for a semilinear initial value problem, *J. Diff. Eq.* **55** (1984), 204–224.

[We1] H. Weyl, Das asymptotisch Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Answendung auf die Theorie der Hohlraumstrahlung), *Math. Ann.* **71** (1912), 441–479.

[We2] H. Weyl, "Gruppentheorie und Quantenmechanik," Wirsel, 1928; Engl. transl., Dover, 1950.

[We3] H. Weyl, The method of orthogonal projection in potential theory, *Duke Math. J.* **7** (1940), 411–440.

[Wi] N. Wiener, The Dirichlet problem, *J. Math. Phys.* **3** (1924), 127–146.

[Wo] W. Wolibner, Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long, *Math. Z.* **37** (1933), 698–726.

[Ya] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

[Yo1] K. Yosida, On the differentiability and the representation of one parameter semi-groups of linear operators, *J. Math. Soc., Japan* **1** (1948), 15–21.

[Yo2] K. Yosida, "Functional Analysis," Springer-Verlag, Berlin, 1965.

[Z-K] N. J. Zabusky and M. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15** (1965), 240–243.

[Ze] E. Zeidler, "Nonlinear Functional Analysis and Its Applications," Vol. I–IV, Springer-Verlag, Berlin, 1985–88.

Printed in Belgium