Directions: Show calculation steps if you want partial credit. There are eight problems, worth 10 points each for the first four and 15 each for the last four.

1. (10 pts.) Consider a fish-tagging experiment, where a pond with 50 fish in total has 10 which are tagged. If a biologist captures 5 fish selected at random without replacement, describe an appropriate sample space and find the probability that exactly 3 tagged fish are captured.

Solution: The sample space consists of all distinct choices of 5 fish selected from the 50 -- could be ordered or unordered.

As unordered, there are \( \binom{50}{5} \) equally likely choices. As ordered, there are \( 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \) such equally likely choices.

We now find probabilities in the unordered case: the choice of 3 tagged fish and 2 untagged fish get multiplied to find the number of ways that exactly 3 tagged fish are selected. This is \( \binom{10}{3} \binom{40}{2} \) in total. So our answer is: \( \frac{\binom{10}{3} \binom{40}{2}}{\binom{50}{5}} \approx 0.04418 \).

For the ordered version, we have \( 10 \cdot 9 \cdot 8 \) ways to pick tagged fish in order, and \( 40 \cdot 39 \) ways to pick the two untagged in order, and then \( \binom{5}{3} \) ways to pick which order the tagged fish came in. So this answer is found differently but the same: \( \frac{10 \cdot 9 \cdot 8 \cdot 40 \cdot 39}{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46} \).

2. (10 pts.) How many ways are there to select an unordered collection of \( k \) objects from a set of size \( n \)? Explain briefly how using multiplication and counting ordered selections yields the result you claimed.

Solution: There are \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \) ways. This comes from the multiplication idea when we view each arrangement in order of \( n \) objects as giving a subset of size \( k \) as the first part and size \( n-k \) as the remaining part, and performing the complete ordering as the choice of the \( k \) elements without order, followed by all orderings of those \( k \) and the remaining \( (n-k) \). Therefore the number of choices becomes the ratio with numerator \( n! \) and the denominator as the product \( k! (n-k)! \).
3. (10 pts.) Dub yatown has its eligible voters self-described as 15 percent Independents, 10 percent Democrats, and 75 percent Republicans. In a recent election, half of all Democrats, two-thirds of all Republicans, and one-third of Independents voted. A voter is chosen at random. If this person did not vote in the recent election, what is the probability that they are a Republican?

Solution: As given as percentages of the total population, there are non-voters as follows: 10 percent Independents, 5 percent Democrats, and 25 percent Republicans. Thus out of 40 percent total non-voters, 25 are Republicans, which gives the conditional probability of \( \frac{25}{40} = .625 \) in decimals.

4. (10 pts.) A student has five blue pens and four red pens in the left pocket of her backpack and three blue pens and three red pens in the right pocket. If she transfers one pen chosen at random from the left pocket to the right, what is the probability that two pens then chosen at random from the right pocket without replacement are both blue?

Solution: This is a conditional probability problem, in that we wish to calculate probability of two blues at end choice, which depends on which color pen was moved. So we can call LB the event that we chose Blue from left at first step and LR as the Red. These are mutually exclusive and occur with probabilities \( P(LB) = \frac{5}{9}, P(LR) = \frac{4}{9} \). As well, once that choice is made, the probability of choosing blues twice at the end, which we call BB, is then found by decomposition:

\[
P(BB) = P(BBLB) + P(BBRB) = \frac{5}{9} \cdot \frac{\binom{4}{2}}{\binom{7}{2}} + \frac{4}{9} \cdot \frac{\binom{3}{2}}{\binom{7}{2}} = \frac{2}{9}.
\]
5. (15 pts.) For the random variable $X$ defined as the number of heads minus the number of tails in 4 coin tosses of a fair coin, describe the possible values, the probability mass function, and the expected value of $X$.

Solution: This problem has a standard binomial distribution with 4 tries and probability 1/2, except that instead of counting heads, we count heads minus tails. Therefore the random variable takes on the values 4, 2, 0, -2, -4 with respective probabilities 1/16, 4/16, 6/16, 4/16, 1/16. The expected value is 0, obtained as the sum

$$4 \cdot \frac{1}{16} + 2 \cdot \frac{4}{16} + 0 \cdot \frac{6}{16} + (-2) \cdot \frac{4}{16} + (-4) \cdot \frac{1}{16}.$$ 

6. (15 pts.) Define the random variable $Y$ as the number of tosses of a fair coin until the second time “heads” appears. What is the probability mass function for $Y$? Hint: If the second head appears on toss $n$, then there is exactly one head in the first $n - 1$ tosses and the last toss is heads.

Solution: With hint, we note that one head in $n - 1$ tries is a standard binomial question with answer: probability is $\binom{n-1}{1} 2^{n-1}$. This is multiplied by $1/2$ for the last head. Therefore the variable $Y$ has

$$P\{Y = n\} = \frac{(n-1)}{2^n}, \ n = 2, 3, \ldots$$
7. (15 pts.) In a mini-lottery game, players pick a pair of distinct numbers out of the numbers from one to ten. A pair is then drawn randomly in the lottery by drawing two balls without replacement from an urn. If your pair matches the chosen pair, you win $40; if not, you lose $1. What is your expected winning or loss?

Solution: I wasn’t clear about ordered selection versus unordered, so I used whatever you did. For unordered, we have a sample space of size \( \binom{10}{2} = 45 \), of which you have one way to win, so you have probability \( \frac{1}{45} \) of winning $40. You have a probability of \( \frac{44}{45} \) of losing $1. Therefore your expected win is \( \frac{40}{45} - \frac{44}{45} = -\frac{4}{45} \) dollars.

For ordered, it is harder to win, so you end up with \( 10 \cdot 9 = 90 \) possible choices, with one win. The expected winnings is again negative (loss) but now it comes out to \( \frac{40}{90} - \frac{89}{90} = -\frac{49}{90} \).

8. (15 pts.) Give a full written argument of why the random variable \( X \) given as the number of successes in four trials where each trial has probability of success \( p \), has expected value \( 4p \). This is a binomial distribution. Note that

\[
P[X = k] = \binom{4}{k} p^k (1 - p)^{4-k}, \quad k = 0 \ldots 4.
\]

Solution: The expected value is the sum:

\[
E[X] = 0 \cdot (1 - p)^4 + 1 \cdot 4 p (1 - p)^3 + 2 \cdot 6 p^2 (1 - p)^2 + 3 \cdot 4 p^3 (1 - p) + 4 \cdot p^4
\]

\[
= 4p \{ (1 - p)^3 + 3p (1 - p)^2 + 3p^2 (1 - p) + p^3 \} = 4p.
\]

using the binomial theorem for \( n = 3 \) to get \( (1 - p + p)^3 = 1 \) in the last step.