1. (10 pts.) Let \( f(x, y, z) = (x + y)^{-2} + \sin(\pi z) \).

Find the gradient vector \( \nabla f(x, y, z) \) at all points and the linear approximation to \( f \) near the point \( x = 1, y = 2, z = -1 \).

\[
\begin{align*}
 f_x &= -2(x+y)^{-3}, \\
 f_y &= -2(x+y)^{-3}, \\
 f_z &= \pi \cos(\pi z)
\end{align*}
\]

So \( \nabla f = -2(x+y)^{-3} \mathbf{i} - 2(x+y)^{-3} \mathbf{j} + (\pi \cos(\pi z)) \mathbf{k} \)

At \( (1, 2, -1) \), \( \nabla f = -2(3)^{-3} \mathbf{i} - 2(3)^{-3} \mathbf{j} + (\pi \cos(-1)) \mathbf{k} \)

\[
f(1, 2, -1) = 3^{-2} + 0 = 1/9
\]

Answer:

\[
\frac{1}{9} - \frac{2}{27}(x-1) - \frac{2}{27}(y-2) - \pi(z+1)
\]
2. (15 pts.) If \( u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \), find \( u_t - u_{xx} \).

\[
\begin{align*}
  u &= (4\pi t)^{-1/2} e^{-x^2/4t} \\
  \frac{\partial u}{\partial t} &= -\frac{1}{2} (4\pi t)^{-3/2} (4\pi) \cdot e^{-x^2/4t} \\
  &\quad + (4\pi t)^{-1/2} \cdot (x^2/4t^2) \cdot e^{-x^2/4t} \\
  \frac{\partial u}{\partial x} &= (4\pi t)^{-1/2} \cdot (-2x/4t) \cdot e^{-x^2/4t} \\
  \frac{\partial^2 u}{\partial x^2} &= (4\pi t)^{-1/2} \left[ -\frac{2}{4t} e^{-x^2/4t} + \frac{16x^2}{16t^2} \cdot e^{-x^2/4t} \right]
\end{align*}
\]

\( u_t - u_{xx} \) is cancellating \( \rightarrow \) Answer: 0

3. (15 pts.) (a) Suppose \( h \) is a function of the variables \( u, v, \) and \( w \), say \( h = g(u, v, w) \) and that each of these three quantities \( u, v, w \) is a function of \( x, y \). Express \( \partial h/\partial x, \partial h/\partial y \) in terms of the derivatives of \( g \) and those of \( u, v, w \). DO NOT PUT THIS ANSWER IN THE BOX, JUST BOX IT IN.

\[
\begin{align*}
  \frac{\partial h}{\partial x} &= \frac{\partial h}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial h}{\partial w} \frac{\partial w}{\partial x} \\
  \frac{\partial h}{\partial y} &= \frac{\partial h}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial h}{\partial w} \frac{\partial w}{\partial y}
\end{align*}
\]

(b) If \( h = \ln(u^2 + v^2 + w^2) \) and \( u = x + 2y, v = 2x - y, w = 2xy \), find \( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \) at the point \( x = y = 1 \). PUT THIS ANSWER IN BOX BELOW.

Using above: \[
\begin{align*}
  \frac{\partial h}{\partial u} &= \frac{2u}{u^2 + v^2 + w^2} \\
  \frac{\partial h}{\partial v} &= \frac{2v}{u^2 + v^2 + w^2} \\
  \frac{\partial h}{\partial w} &= \frac{2w}{u^2 + v^2 + w^2}
\end{align*}
\]

\[
\begin{align*}
  \frac{\partial u}{\partial x} &= 1, \frac{\partial u}{\partial y} = 2 \\
  \frac{\partial v}{\partial x} &= 2, \frac{\partial v}{\partial y} = -1 \\
  \frac{\partial w}{\partial x} &= 2y = 2, \frac{\partial w}{\partial y} = 2x = 2
\end{align*}
\]

\[
\begin{align*}
  u &= 1+2 = 3 \quad \text{so} \quad u^2 + v^2 + w^2 = 9 + 1 + 4 = 14 \\
  v &= 2-1 = 1 \\
  w &= 2\cdot1 = 2
\end{align*}
\]

Answer: \[
\begin{align*}
  h_x &= \frac{6+4\cdot2}{14} = \frac{9}{7} \\
  h_y &= \frac{12-2+8}{14} = \frac{9}{7}
\end{align*}
\]
4. (10 pts.) (a) Find the derivative of the function \( f(x, y, z) = x^2 + y^2 + z^4 \) at \( P_0(-1, 1/2, 1) \) in the direction of the unit vector \( \vec{u} = \frac{-i + 2j - 2k}{3} \).

\[
\nabla f \cdot \vec{u} = \left( 2x \mathbf{i} + 2y \mathbf{j} + 4z^3 \mathbf{k} \right) \cdot \left( \frac{-i + 2j - 2k}{3} \right) \\
= \frac{2(-1) + 2(1/2) - 2}{3} = \frac{-4}{3}
\]

(b) Find the direction(s) in which the same function \( f \) is increasing most rapidly at the same \( P_0 \) and find that rate of increase.

\[ |\nabla f| \text{ is maximum, } \frac{\nabla f}{|\nabla f|} = \text{direction (unit vector)} \]

\[ \text{Ans: } \nabla f = \nabla f|_{P_0} = \sqrt{4 + 1 + 16} = \sqrt{21} \]

\[ \frac{\nabla f}{|\nabla f|} = \frac{-2i + j + 4k}{\sqrt{21}} \]

5. (10 pts.) If \( r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x) \), express the differentials \( dr \) and \( d\theta \) in terms of \( dx \) and \( dy \).

\[
\begin{align*}
\, \, dr &= \frac{1}{2} (x^2 + y^2)^{-1/2} \cdot 2x \, dx + \frac{1}{2} (x^2 y^2)^{-1/2} \cdot 2y \, dy \\
&= \frac{x}{\sqrt{x^2 + y^2}} \, dx + \frac{y}{\sqrt{x^2 + y^2}} \, dy \\
\, \, d\theta &= \frac{1}{1 + (\frac{y}{x})^2} \cdot (-\frac{y}{x^2}) \, dx + \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{1}{x} \, dy \\
&= -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy \\
\end{align*}
\]

\[ \text{Ans: } \]
6. (15 pts.) (a) What are critical points of a function $f(x,y)$?

Points where $\nabla f = 0$ or does not exist

(b) Describe briefly why critical points are the only candidates for local maxima or minima (Hint: what is excluded and why?)

If $\nabla f$ exists and is non-zero, there is an uphill ($\nabla f$) direction and a downhill ($-\nabla f$) locally, so it can't be a local max/min.

(c) How do you decide if a critical point is a local maximum or minimum?

Second derivative test: Look at $f_{xx} f_{yy} - f_{xy}^2$ at the pt. (and at $f_{xx}$ also)

$f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$: Local min

$f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$: Local max

$f_{xx} f_{yy} - f_{xy}^2 < 0$: Saddle point
7. (15 pts.) Find all the local maxima, local minima, and saddle points of the function \( f(x, y) = (y^2/2) + \cos(x) \) for \(-\pi/2 \leq x \leq 3\pi/2\).

\[
\begin{align*}
fx &= -\sin x & \text{critical pt.: } y=0 \\
fy &= 2y
\end{align*}
\]

\[
\begin{align*}
\text{Second derr test:} \quad fx &= -\cos x : \quad fx(0,0) = -1 \\
fy &= 2 \\
fxy &= 0 \\
fxx(\pi,0) &= 1 \\
fyy(\pi,0) &= 2 \\
fxy(\pi,0) &= 0
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
\text{Local min} & \quad \text{at } (0,0) \\
\text{no local maxima}
\end{aligned}
\end{align*}
\]

Ans:
8. (15 pts.) (a) How are level sets related to the gradient vector field?

\[ \nabla f \text{ is orthogonal to level sets} \text{ [direction]} \]

\[ \text{also, } |\nabla f| \text{ is related to spacing of level sets} \]

- closer levels \( \Rightarrow \) larger gradient \( \text{[magnitude]} \)

(b) Given the level sets picture below, sketch the gradient field on it paying attention to where the gradient is large and where it is small and to its direction. Include at least points marked by dots.

\[ f=16 \quad f=14 \quad f=12 \quad f=10 \quad f=8 \quad f=6 \quad f=4 \quad f=2 \quad f=1 \]