NAME ANSWER KEY

Math 125-B01, Summer 2014, Test 3, O'Beirne

				Value
1	a		4, 4	16
	b		4, 4	
2	a		~~8~~ 9	16
	b		~~8~~ 9	
3	(i)		8	16
	(ii)		8	
4	a	(i)	3	18
		(ii)	3	
		(iii)	3	
	b	(i)	3	
~~5~~		(ii)	3	
		(iii)	3	
~~6~~ 5	a		8	16
	b		8	
6	a		4	16
	b		4	
	c		4	
	d		4	

Answer all questions. You must show appropriate work leading to your answer for full credit. You must work alone

1. (a) Given the arithmetic sequence with first term 9 and common difference -2/3.

What is the 42nd term? $-18\frac{1}{3}$

What is the sum of the first 27 terms? 9

Work:

$$9 + 41\left(-\frac{2}{3}\right) = -18\frac{1}{3}$$

$$\frac{27}{2}\left(2\cdot 9 + 26\left(-\frac{2}{3}\right)\right) = 9$$

(b) Given the geometric sequence with first term 1,000 and common ratio 1.05.

What is the 17th term of the sequence? $1000(1.05)^{16} = 2,182.87$

What is the sum of the first 20 terms? $\dfrac{1000[1-1.05^{20}]}{1-1.05} = 33,065.95$

Work:

$$a_{17} = ar^{16} = 1000(1.05)^{16}$$

$$S_{20} = \frac{a - ar^{20}}{1-r} = \frac{1000 - 1000(1.05)^{20}}{1-1.05} = \frac{1000[1-1.05^{20}]}{1-1.05}$$

2. (a) Solve the following recurrence relation explicitly for a_n

$a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 3$ $\qquad a_n =$ $2^{n+1} - 1$ (6)

What is a_{15}? $2^{16} - 1 = 65{,}535$ (3)

Work:

Check

$1, 3, 7, 15, \ldots$

$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x_1 = 1$$
$$x_2 = 2$$

$$a_n = c_1 1^n + c_2 2^n$$

$n=0$: $a_0 = 1 = c_1 + c_2 \rightarrow c_1 = -1$

$n=1$: $a_1 = 3 = c_1 + 2c$ $\qquad c_2 = 2$

(b) Solve the following recurrence relation explicitly for a_n

$a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 2$ and $a_1 = 6$ $\qquad a_n$ $2^{n+1} + n2^n = 2^n(n+2)$ (6)

What is a_{15}? $2^{15} \cdot 17 = 557{,}056$. (3)

Work:

$$x^2 - 4x + 4 = 0$$
$$(x-2)^2$$
$$x = 2$$

$$a_n = c_1 2^n + c_2 n 2^n = 2 \cdot 2^n + 1 \cdot n \cdot 2^n = 2^{n+1} + n2^n$$
$$= 2^n(2+n)$$

$n=0$: $a_0 = 2 = c_1 + 0 \rightarrow c_1 = 2$

$n=1$: $a_1 = 6 = 2c_1 + 2c_2$ $\qquad c_2 = 1$

Check: $2, 6, 16, 40$

3. How many of the integers between 1 and 800 (inclusive)…

(i) are divisible by 2 or 3 or 5? $\underline{586}$

Work:

$\left\lfloor \frac{800}{2} \right\rfloor = 400$ $\quad \left\lfloor \frac{800}{6} \right\rfloor = 133$

$\left\lfloor \frac{800}{3} \right\rfloor = 266$ $\quad \left\lfloor \frac{800}{10} \right\rfloor = 80$ $\quad \left\lfloor \frac{800}{30} \right\rfloor = 26$

$\left\lfloor \frac{800}{5} \right\rfloor = 160$ $\quad \left\lfloor \frac{800}{15} \right\rfloor = 53$

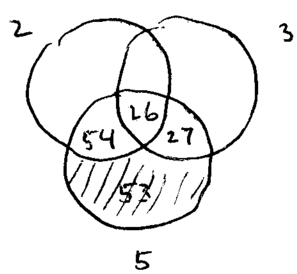
$$|A_2 \cup A_3 \cup A_5| = 400 + 266 + 160 - 133 - 80 - 53 + 26 = 586$$

(Prin of Incl/Excl)

(ii) are divisible by 5 but not by 2 or 3? $\underline{53}$

Work:

From above:



4. (a) How many numbers in the range 100-799 (inclusive)…

(i) have no repeated digits? ~~[illegible]~~ 504

Work: ~~[illegible]~~ ~~[illegible]~~

$\underline{7} \cdot \underline{9} \cdot \underline{8} = 504$

(ii) are odd and have no repeated digits? 248

Work:

Case 1: First digit odd

$4 \times 8 \times 4 = 128$

Case 2: First digit even

$3 \times 8 \times 5 = 120$

248

(iii) are even and have no repeated digits? 256

Work:

$504 - 248 = 256$

(b) How many 4-digit numbers are there from 1000 to 9999 (inclusive)

(i) if repetitions are allowed? 9,000

Work:

$\underline{9} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 9000$

(ii) if repetitions are not allowed? 4,536

(3) Work

81
56
486
405
4536

$\overline{9}\ \overline{9}\ \overline{8}\ \overline{7}$

(iii) if one or more digits are repeated? 4,464

(3) Work:

9000 − 4536 = 4464

5. (a) The Pigeonhole Principle shows that in any sequence of 10 natural numbers there is a "string" of consecutive terms whose sum is divisible by 10. In the

(8) following sequence circle that string.

40

2 4 3 5 7 7 5 4 9 9

Eq Class of Partial Sums $\bar{2}$ $\bar{6}$ $\bar{9}$ $\bar{4}$ $\bar{1}$ $\bar{8}$ $\bar{3}$ $\bar{7}$ $\bar{6}$ $\bar{5}$

(b) A is a set containing seven different natural numbers, each less than or equal to 21. Use the Pigeonhole Principle to show that if the elements in each non-empty

(8) subset of A are summed, the total for at least two of the non-empty subsets must be equal. (Hint: How many non-empty subsets are there?)

Work:

A has $2^7 - 1$ non-empty subsets $2^7 - 1 = 127$.

Largest sum of 7 naturals each diff and ≤ 21 is
$21+20+19+18+17+16+15 = 126$

Smallest non-empty set count is 1
$\{1\}$
Largest is 126
$\{15, 16, \ldots, 21\}$

If you assign 127 sets to 126 numbers then at least 2 sets must have the same number (PP) (Pigeonhole Principle)

6. Suppose you take a test and you must answer exactly 9 of 12 questions

(a) In how many ways can you choose the questions you answer? (The order doesn't matter) (4)

220

Work:

$$\binom{12}{9} = \binom{12}{3} = \frac{\cancel{12}^{2} \cdot 11 \cdot 10}{\cancel{3} \cdot \cancel{2} \cdot 1} = 220$$

(b) In how many ways can you choose 9 questions if you must answer all four of the first questions?

56 (4)

Work: Choose 5 of last 8

$$\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 56$$

(c) In how many ways can you choose 9 if you must answer at least three of the last six questions and at most three of the first four questions?

164 (4)

Work:

6 Cases:

First 4	Middle 2	Last 6		
3	2	4	= 9	$\binom{4}{3}\binom{2}{2}\binom{6}{4} = \binom{4}{1}\binom{2}{2}\binom{6}{2} = 4 \cdot 1 \cdot 15 = 60$
2	2	5	= 9	$\binom{4}{2}\binom{2}{2}\binom{6}{5} = 6 \cdot 1 \cdot 6 = 36$
3	1	5	= 9	$\binom{4}{3}\binom{2}{1}\binom{6}{5} = 4 \cdot 2 \cdot 6 = 48$
1	2	6	= 9	$\binom{4}{1}\binom{2}{2}\binom{6}{6} = 4 \cdot 1 \cdot 1 = 4$
2	1	6	= 9	$\binom{4}{2}\binom{2}{1}\binom{6}{6} = 6 \cdot 2 \cdot 1 = 12$
3	0	6	= 9	$\binom{4}{3}\binom{2}{0}\binom{6}{6} = 4 \cdot 1 \cdot 1 = 4$

} 164

(d) In how many ways can you choose 9 if you must answer at least four of the last six questions?

200 (4)

Work:

First 6	Last 6	
5	4	$\binom{6}{5}\binom{6}{4} = 6 \cdot 15$
4	5	$\binom{6}{4}\binom{6}{5} = 15 \cdot 6$
3	6	$\binom{6}{3}\binom{6}{6} = 20 \cdot 1$

} 200