Answer all questions. Show the work leading to the answer under each question for credit, if indicated. The Honor Code is in effect. You must work alone.

(16) 1. Test the validity of the following argument. If it is a valid argument show why. If it is not a valid argument specify truth values for p, q, and r that produce true premises but a false conclusion.

$$\begin{array}{l} p \vee (q \rightarrow r) \\ q \vee r \\ r \rightarrow p \\ \hline \neg p \vee q \end{array}$$

Work: INVALID

Use Truth Table to see that when p is T, q is F, r is T the premises are T but conclusion is F

(6) 2. (a) If $A \cup B = A \cup C$ does $B = C$? Explain fully. No

Work: Counterexample

Let $A = \{1, 2\}$ $B = \{1\}$ $C = \{2\}$

Then $A \cup B = A \cup C$ but $B \neq C$

(6) (b) If $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$ does $B = C$? Explain fully. Yes

Work:

$B = (A \cap B) \cup (A^c \cap B) = (A \cap C) \cup (A^c \cap C) = C$ So $B = C$

3. Given the following tables which depict relations. For each one, circle whether the relation is Reflexive (R), Symmetric (S), Transitive (T), or Anti-Symmetric (A-S). Also circle whether each represents an Equivalence Relation (ER) or a Partial Order (PO). (12)

	a	b	c	d
a	X		X	X
b		X		
c	X		X	
d		X		X

	a	b	c	d
a	X			
b		X		
c			X	X
d			X	X

	a	b	c	d
a	X	X		
b		X		
c	X	X	X	X
d	X	X		X

	a	b	c	d
a	X	X	X	X
b				X
c				X
d				X

Table 1	Table 2	Table 3	Table 4
R	R	R	R
S	S	S	S
T	T	T	T
A-S	A-S	A-S	A-S
ER	ER	ER	ER
PO	PO	PO	PO

4. (a) Express in disjunctive normal form: $p \vee ((\neg p) \wedge q)$ (6)

$(p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge q)$

Work:

(b) Given that $287^2 + 1 = 2(41{,}185)$ express 41,185 as the sum of two squares

$41{,}185 = 143^2 + 144^2$ (6)

Work:

$m^2 + 1 = 2m$

m^2 odd $= 2k+1$

$\rightarrow k = 143$

$(2k+1)^2 + 1 = 2m$

$4k^2 + 4k + 2 = 2m$

$m = 2k^2 + 2k + 1$

$= (k)^2 + (k+1)^2$

$\therefore\ 143^2 + 144^2 = 41{,}185$

(c) Assuming k ≤ d list the elements of the set {dkk, d, kk, dk, ddk, kd, dkkd, kdk} in lexicographic order. (4)

kk, kd, kdk, d, dk, dkk, dkkd, ddk

(16) 5. (a) **Proposition 2.4.3**: Let $\sim$ be an equivalence relation on A. For any $x \varepsilon A$, $x \sim a$ if and only if sets $\bar{x} = \bar{a}$.

Proof (fill in the blanks with the reason for that step):

First show that sets $\bar{x} = \bar{a} \rightarrow x \sim a$. Reason

$\bar{x} = \bar{a}$	
$x \in \bar{x}$	
$x \in \bar{a}$	def of equiv class/reflexive
Therefore, $x \sim a$	

Now show that $x \sim a \rightarrow \bar{x} = \bar{a}$

$x \sim a$	
Let $y \in \bar{x}$	
$y \sim x$	
$x \sim a$	
$y \sim a$	transitivity
$y \in \bar{a}$	
Therefore $\bar{x} \subseteq \bar{a}$	def of a subset

$x \sim a$	
Now let $y \in \bar{a}$	
$y \sim a$	
$a \sim x$	symmetry
$y \sim x$	
$y \in \bar{x}$	def of equiv class
Therefore $\bar{a} \subseteq \bar{x}$	
Therefore $\bar{x} = \bar{a}$	def of set equality.

(b) **Proposition 2.4.4**: Let $\sim$ be an equivalence relation on set A. Let $a, b \varepsilon A$. Then sets $\bar{a}$ and $\bar{b}$ are either equal or are disjoint.

Proof (fill in the reasons):

If sets $\bar{a}$ and $\bar{b}$ are equal the proof is complete. So assume they are not equal.

Now assume sets $\bar{a}$ and $\bar{b}$ are not disjoint i.e. $\bar{a} \cap \bar{b} \neq \emptyset$

Let $x \in \bar{a} \cap \bar{b}$	Assumed
$x \in \bar{a}$	def of set intersection
$x \sim a$	
$\bar{x} = \bar{a}$	Prop 2.4.3
$x \in \bar{b}$	
$x \sim b$	
$\bar{x} = \bar{b}$	
Therefore $\bar{a} = \bar{b}$	

Contradiction. Therefore, they are either equal or disjoint.

6. Define a ~ b if and only if $a^2 - b^2$ is evenly divisible by 3, for a, b in the set of Integers **Z**.

(a) Show that ~ is an equivalence relation on **Z** (8)

Work:

R: $a \sim a$ since $a^2 - a^2 = 0 = 3k \quad k \in \mathbb{Z}$

S: if $a \sim b$ then $a^2 - b^2 = 3k$

$\rightarrow b^2 - a^2 = 3(-k) \quad -k \in \mathbb{Z}$

T: if $a \sim b$ and $b \sim c$ then

$$\begin{array}{l} a^2 - b^2 = 3k \\ b^2 - c^2 = 3l \\ \hline a^2 - c^2 = 3(k+l) \end{array} \quad k+l \in \mathbb{Z}$$

(b) Show the equivalence classes created by ~ (4)

Work:

$\bar{0} = \{0, \pm 3, \pm 6, \pm 9, \ldots\} = 3\mathbb{Z}$

$\bar{1} = \{\ldots -5, -4, -2, -1, 1, 2, 4, 5, \ldots\}$ $3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$

(c) Show the quotient set **Z** mod ~ (4)

$\mathbb{Z}/\sim \; = \{\bar{0}, \bar{1}\}$

7. Given the set A = {4, 6, 8, 12, 18, 24}. For a, b define "$\le$" as: $a \le b$ if a evenly divides b.

(6) (a) Does this "$\le$" define a partial order on A? Explain. Yes

Work:

	4	6	8	12	18	24
4	x		x	x		x
6		x		x	x	x
8			x			x
12				x		x
18					x	
24						x

∴ R ✓

A-S ✓

T ✓

$a \le a$ because $\frac{a}{a} = 1 \in$

if $\frac{a}{b} = k \in \mathbb{Z}$ $a = bk$

$\frac{b}{a} = \ell \in \mathbb{Z}$ $b = a\ell$

$a = a\underbrace{\ell k}_{1}$

$\rightarrow a = b$

if $\frac{a}{b} = k$ $\frac{b}{c} = \ell$ $k, \ell \in$

then $\cancel{\frac{\frac{a}{b}}{\frac{b}{c}} = \frac{a}{c} = \frac{k}{\ell}}$

$\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} = k\ell$

$a \sim c$

(6) (b) Draw the Hasse Diagram for (A, $\le$)

Work: