MATH 325, Section 001
Spring, 2011
Exam 3 Name Solutions

1. Suppose that the sequences (a_n) and (b_n) satisfy the recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + b_{n-1} \\ b_n &= a_n + a_{n-1} + b_{n-1} \end{aligned}$$

with initial conditions $a_0 = 0$ and $a_1 = 1$.

(a) Find a single recurrence relation (u_n) which is satisfied by both (a_n) and (b_n).

We look for a recurrence relation of the form

$$u_n = Cu_{n-1} + Du_{n-2}.$$

From the initial conditions and the first equation, we get $1 = 3 \cdot 0 + b_0$, so

$$b_0 = 1.$$

Therefore, from the second equation,

$$b_1 = 1 + 0 + 1 = 2.$$

We can now get

$$a_2 = 3 \cdot 1 + 2 = 5$$

and

$$b_2 = 5 + 1 + 2 = 8.$$

We now substitute these values into the equation for u_n with $n = 2$ to get

$$\begin{aligned} 5 &= C \\ 8 &= 2C + D. \end{aligned}$$

Solving gives $C = 5$ and $D = -2$, so

$$u_n = 5u_{n-1} - 2u_{n-2}.$$

(b) Use mathematical induction to show that each of the sequences (a_n) and (b_n) satisfies the recurrence relation which you found.

If $n = 2$, $a_n = 5 = 5\cdot 1 - 0 = 5a_1 - 2a_0 = 5a_{n-1} - 2a_{n-2}$ and $b_n = 8 = 5\cdot 2 - 2 = 5b_1 - 2b_0 = 5b_{n-1} - 2b_{n-2}$. Therefore, both (a_n) and (b_n) satisfy the recurrence relation for $n = 2$. Now suppose $n > 2$ and that for all k such that $2 \le k < n$, a_k and b_k satisfy the recurrence. Then

$$\begin{aligned} a_n &= 3a_{n-1} + b_{n-1} \\ &= 3(5a_{n-2} - 2a_{n-3}) + (5b_{n-2} - 2b_{n-3}) \\ &= 5(3a_{n-2} + b_{n-2}) - 2(3a_{n-3} + b_{n-3}) \\ &= 5a_{n-1} - 2a_{n-2} \end{aligned}$$

and

$$\begin{aligned} b_n &= a_n + a_{n-1} + b_{n-1} \\ &= (5a_{n-1} - 2a_{n-2}) + (5a_{n-2} - 2a_{n-3}) + (5b_{n-2} - 2b_{n-3}) \\ &= 5(a_{n-1} + a_{n-2} + b_{n-2}) - 2(a_{n-2} + a_{n-3} + b_{n-3}) \\ &= 5b_{n-1} - 2b_{n-2}. \end{aligned}$$

2. In the mythical country of Hedfruskl, the unit of currency is the zliptwyc. Find the generating function $g(x)$ for the number ways are there of making change for a 100 zliptwyc bill using only 1 zliptwyc, 3 zliptwyc, 7 zliptwyc, and 10 zliptwyc bills. Express your answer as a rational function.

Let e_1, e_2, e_3, and e_4 be the number of 1, 3, 7, and 10 zliptwyc notes respectively. Then we want the number of non-negative integer solutions of

$$e_1 + 3e_2 + 7e_3 + 10e_4 = 100.$$

Let $d_1 = e_1$, $d_2 = 3e_2$, $d_3 = 7e_3$, and $d_4 = 10e_4$. We want the number of non-negative integer solutions of

$$d_1 + d_2 + d_3 + d_4 = 100$$

such that d_2 is divisible by 3, d_3 is divisible by 7, and d_4 is divisible by 10. Therefore,

$$\begin{aligned} g(x) &= (1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^7 + x^{14} + \cdots)(1 + x^{10} + x * 20 + \cdots) \\ &= (\sum_{n=1}^{\infty} x^n)(\sum_{n=1}^{\infty} x^{3n})(\sum_{n=1}^{\infty} x^{7n})(\sum_{n=1}^{\infty} x^{10n}). \end{aligned}$$

Using the formula for the sum of a geometric series gives

$$g(x) = (\frac{1}{1-x})(\frac{1}{1-x^3})(\frac{1}{1-x^7})(\frac{1}{1-x^{10}}).$$

Therefore,

$$g(x) = \frac{1}{(1-x)(1-x^3)(1-x^7)(1-x^{10})}.$$

3. Use generating functions to find the number of non-negative integer solutions of

$$e_1 + e_2 = 100$$

such that e_1 is even and e_2 is at least 1. [You must use generating functions to do this problem, although after you get your answer, you might try to think of another way of doing the problem to verify that your answer is correct.]

The generating function is given by

$$
\begin{aligned}
g(x) &= (1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots) \\
&= \frac{1}{1-x^2}\frac{x}{1-x} \\
&= \frac{x}{(1-x)^2(1+x)}.
\end{aligned}
$$

The partial fraction expansion for $g(x)$ has the form

$$g(x) = \frac{x}{(1-x)^2(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x}.$$

Therefore,

$$x = A(1-x^2) + B(1+x) + C(1-x)^2.$$

This equation can be written as

$$(C-A)x^2 + (B-2C)x + (A+B+C) = x.$$

Comparing coefficients gives the system

$$
\begin{aligned}
A - C &= 0 \\
B - 2C &= 1 \\
A + B + C &= 0.
\end{aligned}
$$

Solving this system gives $A = C = -\frac{1}{4}$ and $B = \frac{1}{2}$. Therefore,

$$g(x) = -\frac{1}{4}\left(\frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{(1-x)^2}\right) - \frac{1}{4}\left(\frac{1}{1+x}\right).$$

We want the coefficient of x^{100} in this expansion. Using the formula for the sum of a geometric series and the generalized binomial theorem gives that the coefficient of x^{100} is $-\frac{1}{4}+\frac{1}{2}\binom{-2}{100} - \frac{1}{4}$. Since $\binom{-2}{100} = 101$, the coefficient of x^{100} is 50, so there are 50 solutions.

4. Use generating functions to solve the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 0$. The formula $\binom{-2}{k} = (-1)^k(k+1)$ might be helpful for this problem. [You have to do this with generating functions–no credit will be given unless it is clear that that is how you are doing the problem.]

`Multiplying the given equation by` x^n `gives`

$$a_n x^n = 6a_{n-1}x^n - 9a_{n-2}x^n.$$

`Let` $A(x) = \sum_{n=2}^{\infty} a_n x^n$. `Then`

$$\begin{aligned}
A(x) &= 6\sum_{n=2}^{\infty} a_{n-1}x^n - 9\sum_{n=2}^{\infty} a_{n-2}x^n \\
&= 6x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 9x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} \\
&= 6x\sum_{n=1}^{\infty} a_n x^n - 9x^2\sum_{n=0}^{\infty} a_n x^n \\
&= 6x(a_1x + A(x)) - 9x^2(a_0 + a_1x + A(x)) \\
&= 6xA(x) - 9x^2(1 + A(x)).
\end{aligned}$$

`Solving for` $A(x)$ `gives`

$$A(x) = \frac{-9x^2}{(1-3x)^2}.$$

`From the generalized binomial theorem,`

$$\begin{aligned}
\frac{1}{(1-3x)^2} &= \sum_{n=0}^{\infty}\binom{-2}{n}(-3x)^n \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)(-3)^n x^n \\
&= \sum_{n=0}^{\infty}(n+1)3^n x^n.
\end{aligned}$$

Therefore,

$$A(x) = \frac{-9x^2}{(1-3x)^2} = -9x^2 \sum_{n=0}^{\infty}(n+1)3^n x^n,$$

so

$$A(x) = -\sum_{n=0}^{\infty}(n+1)3^{n+2}x^{n+2} = -\sum_{n=2}^{\infty}(n-1)3^n x^n,$$

that is,

$$A(x) = \sum_{n=2}^{\infty}(1-n)3^n x^n.$$

Therefore, $a_n = (1-n)3^n$ for $n = 2, 3, \cdots$.

5. Suppose that the sequences (a_n) and (b_n) satisfy the recurrence relations

$$\begin{aligned} a_n &= a_{n-1} + b_{n-1} \\ b_n &= a_n + a_{n-1} \end{aligned}$$

with initial conditions $a_0 = 0$ and $b_0 = 1$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating functions for (a_n) and (b_n) respectively. Express $A(x)$ and $B(x)$ as rational functions.

$A(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} b_{n-1}x^n = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} \infty b_n x^{n+1}$. From the initial conditions, we get $\sum_{n=0}^{\infty} \infty a_n x^{n+1} = xA(x)$ and $\sum_{n=0}^{\infty} b_n x^{n+1} = x + xB(x)$. Therefore,

$$A(x) = xA(x) + xB(x).$$

$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + \sum_{n=0}^{\infty} b_n x^n = 1 + \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 1 + A(x) + xA(x)$. Therefore,

$$B(x) = 1 + (1+x)A(x).$$

Substituting in the equation $A(x) = xA(x) + xB(x)$ and simplifying gives

$$A(x) = \frac{x}{1-2x-x^2},$$

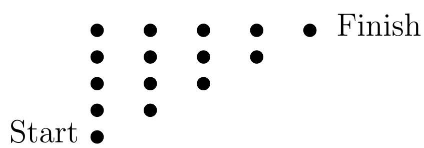
and substituting this into the equation $B(x) = 1+(1+x)A(x)$ gives

$$B(x) = \frac{1-x}{1-2x-x^2}.$$

6. Find the exponential generating function for the number a_n of n-permutations of the multiset $\{\infty \cdot A, \infty \cdot B, \infty \cdot C\}$ such that there are an even number of As. Express your answer in terms of exponential functions.

The exponential generating function is $g^{(e)}(x) = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots)(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots) = (\cosh x)e^x e^x = (\frac{1}{2})(e^x+e^{-x})e^{2x} = (\frac{1}{2})(e^{3x} + e^x)$.

7. In the pattern below, how many ways are there to go from the dot on the bottom to the dot on the top right if each step consists of one step up or one stem to the right?



The problem asks for the number of sequences of 4 U's and 4 R's (U for 'Up' and R for 'Right') so that the number of R's to the left of a U does not exceed the number of U's. That number is the Catalan number $C_4 = \frac{1}{4+1}\binom{2 \cdot 4}{4} = (\frac{1}{5})70 = 14$.