MATH 325, Section 001
Spring, 2011
Exam 1 Name `Solutions`

1. How many ways are there of arranging three boxing matches from a group of 10 Democrats and 7 Republicans if each match is to consist of a Democrat against a Republican? Assume that the order of the matches does not matter, and that nobody is in more than one match.

   `First choose` 3 `Democrats to participate in the matches. There are` $\binom{10}{3}$ `ways of doing this. Arrange the three Democrats in a row. Then choose` 3 `Republicans to participate in the matches and arrange them to determine which Democrat each will box. There are` $P(7,3)$ `ways of doing this. By the Multiplication Principle, there are`

$$P(7,3)\binom{10}{3} = \frac{7!}{4!}\frac{10!}{3!7!} = \frac{10!}{3!4!}$$

   `ways of setting up the matches.`

Answer:
$\frac{10!}{3!4!}$

2. How many ways are there of arranging three boxing matches from a group of 17 Republicans? Assume that the order of the matches does not matter, and that nobody is in more than one match.

   `First select the` 6 `people who will participate in the matches. There are` $\binom{17}{6}$ `ways of doing that. Arrange the six people in a row. Select an opponent for the first person in the row. There are` 5 `ways of doing that. There are now` 4 `people in the row who do not have opponents. Select an opponent for the first of these four people. There are` 3 `ways of doing that. This leaves` 2 `people who will fight each other. By the Multiplication Principle there are`

$$\binom{17}{6}\cdot 5\cdot 3 = 15\binom{17}{6} = \frac{15\cdot 17!}{6!11!}$$

   `ways of setting up the matches.`

Answer:
$\frac{15\cdot 17!}{6!11!}$

3. Find the number of 12-bit strings of 0's and 1's having exactly 4 0's no two of which are adjacent.

There will be 8 1's. We place the 8 1's in a row with spaces between them and at the beginning and end:

$$-1-1-1-1-1-1-1-1-.$$

We choose 4 of the 9 spaces in which to put the 4 0's. There are $\binom{9}{4} = \frac{9!}{4!5!}$ ways of doing this.

Answer:

$$\frac{9!}{4!5!}$$

4. How many ways are there of assigning 4 A's, 5 B's, 8 C's, and 2 D's to a class of 19 students?

View the assignment of grades as an arrangement of the 19-letter ''word'' AAAABBBBBCCCCCCCCDD where the first student on the class list gets for a grade the first letter in the arrangement, the second student on the class list gets for a grade the second letter in the arrangement, and so on. Therefore, we want the number of 19-permutations of the multiset $\{4 \cdot A, 5 \cdot B, 8 \cdot C, 2 \cdot D\}$. There are

$$\binom{19}{4,5,8,2} = \frac{19!}{4!5!8!2!}$$

such 19-permutations.

Answer:

$$\frac{19!}{4!5!8!2!}$$

5. How many ways are there of choosing three scoops of ice-cream to be used in a banana split if there are 31 flavors from which to choose. (The flavors can be mixed in any way, so, for example, it is possible to get two scoops of vanilla and one scoop of tutti-frutti. Assume that the order in which the flavors are chosen does not matter.)

`This problem asks for the number of` 3`-combinations of a multiset having` 31 `kinds of objects each having repetition number` ∞ `(or at least` 3`). There are` $\binom{3+31-1}{3} = \binom{33}{3} = \frac{33!}{3!30!}$ `such` 3`-combinations.`

Answer:

$\frac{33!}{30!3!}$

6. How many integer solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

such that $x_k \geq 0$ for $k = 1, 2, 3, 4$?

`We want the number of` 15`-combinations of` 15`-combinations of the multiset` $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4\}$`, which is`

$$\binom{4+15-1}{15} = \binom{18}{15} = \frac{18!}{15!3!}.$$

Answer:

$\frac{18!}{15!3!}$

7. How many integer solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 15$$

such that $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 3$, and $1 \leq x_4 \leq 4$?

First we let $y_2 = x_2 - 1$, $y_3 = x_3 - 3$, and $y_4 = x_4 - 1$. Then we want the number of solutions of $x_1 + y_2 + y_3 + y_4 = 10$ such that $0 \leq x_1$, $0 \leq y_2$, $0 \leq y_3$, and $0 \leq y_4 \leq 3$. The number of solutions of $x_1 + y_2 + y_3 + y_4 = 10$ such that $0 \leq x_1$, $0 \leq y_2$, $0 \leq y_3$, and $0 \leq y_4$ is $\binom{4+10-1}{10} = \binom{13}{10}$. Now let $z_4 = y_4 - 4$. The number of solutions of $x_1 + y_2 + y_3 + y_4 = 10$ such that $0 \leq x_1$, $0 \leq y_2$, $0 \leq y_3$, and $3 < y_4$ is the same as the number of solutions of $x_1 + y_2 + y_3 + z_4 = 6$ such that $0 \leq x_1$, $0 \leq y_2$, $0 \leq y_3$, and $0 \leq z_4$, which is $\binom{6+4-1}{6} = \binom{9}{6}$. Therefore, the number of solutions of the original problem is

$$\binom{13}{10} - \binom{9}{6} = \frac{13!}{10!3!} - \frac{9!}{6!3!}.$$

Answer:
$\frac{13!}{10!3!} - \frac{9!}{6!3!}$

8. Find the first three terms in the series representation of $\frac{1}{(1-x)^3}$, that is, if $\frac{1}{(1-x)^3} = c_0 + c_1 x + c_2 x^2 + \cdots$, find $c_0, c_1 x,$ and $c_2 x^2$.

By the Binomial Theorem,

$$(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{-3}{k} (-1)^k x^k,$$

so

$$c_0 = \binom{-3}{0} = 1,$$

$$c_1 = -\binom{-3}{1} = -\frac{-3}{1!} = 3,$$

and

$$c_2 = \binom{-3}{\ } = \frac{(-3)(-4)}{2!} = 6.$$

$c_0 =$	1
$c_1 x =$	$3x$
$c_2 x^2 =$	$6x^2$

9. Show that if n is a positive integer, then

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{(n-k)}.$$

Proof:

$3^n = (2+1)^n$, so by the Binomial Theorem,

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

On the other hand, $3^n = (4-1)^n$, so applying the Binomial Theorem gives

$$3^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{(n-k)}.$$

Comparing the two expressions for 3^n gives the result.

10. How many ways are there of placing three red rooks, three blue rooks, and two yellow rooks on an eight-by-eight chess board in such a way that no two rooks are in the same row or the same column.

There are $8!$ ways of choosing the positions in which the 8 rooks will be placed. Once the positions are chosen, the number of ways of arranging the rooks is the number of permutations of the multiset $\{3 \cdot \texttt{red}, 3 \cdot \texttt{blue}, 2 \cdot \texttt{yellow}\}$, and there are $\binom{8}{3,3,2} = \frac{8!}{3!3!2!}$ such permutations. Therefore, by the Multiplication Principle there are $\frac{(8!)^2}{3!3!2!}$ ways of placing the rooks.

Answer:

$\frac{(8!)^2}{3!3!2!}$

11. Suppose that k and n are integers such that $2 \leq k \leq n$. Prove that $\binom{n}{k}+2\binom{n}{k-1}+\binom{n}{k-2} = \binom{n+2}{k}$. [Hint: Use Pascal's Identity.]

Proof:

Applying Pascal's Identity several times gives $\binom{n}{k} + 2\binom{n}{k-1} + \binom{n}{k-2} = (\binom{n}{k} + \binom{n}{k-1}) + (\binom{n}{k-1} + \binom{n}{k-2}) = \binom{n+1}{k} + \binom{n+1}{k-1} = \binom{n+2}{k}$.

12. Find the coefficient of x^3y^2z in the expansion of $(x + 2y + 3z)^6$.

By the Multinomial Theorem, the coefficient of $x^3(2y)^2(3z)$ is $\binom{6}{3,2,1} = \frac{6!}{3!2!1!}$. Therefore, the coefficient of x^3y^2z is $2^2 \cdot 3 \cdot \frac{6!}{3!2!1!} = 6!$.

Answer:

6!