MATH 301, Section A01 Name SOLUTIONS
Summer, 2011
Exam 1 Student ID number________________

PART A. (8 points each) *Carefully* define each of the following terms. Write in complete, grammatically correct sentences. *Do not include anything extraneous, such as related examples.*

1. $a|b$, that is, a divides b, where a and b are integers and $a \neq 0$.

   $a|b$, where $a \neq 0$ if and only if there exists a $k \in \mathbb{Z}$ such that $b = ak$.

2. The Division Algorithm.

   If a is an integer and b is a positive integer, then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.

3. Prime number.

   The positive integer n is prime if $n \neq 1$ and the only divisors of $\pm n$ are n and ± 1.

4. The greatest common divisor of two positive integers a and b.

   The greatest common divisor of a and b, where a and b are positive integers, is the largest integer n such that $n|a$ and $n|b$.

5. The Fundamental Theorem of Arithmetic.

   Every integer $n \geq 2$ can be written in exactly one way as a product of a non-decreasing sequence of finitely many prime numbers.

PART B. (10 points each) *Carefully* do six of the following problems. Mark clearly the problems which you do not want graded, and put in the box below the number of the problem that you do not want graded:

Problems which you don't want graded:

1. (a) Find $\sum_{i=1}^{4}\sum_{j=0}^{2} i^j$.

$\sum_{i=1}^{4}\sum_{j=0}^{2} i^j = \sum_{i=1}^{4}(1+i+i^2) = (1+1+1)+(1+2+4)+(1+3+9)+(1+4+16) = 44.$

Answer:
44

(b) Find $\prod_{k=1}^{4}(2k-3)$

$\prod_{k=1}^{4}(2k-3) = (2\cdot 1-3)(2\cdot 2-3)(2\cdot 3-3)(2\cdot 4-3) = (-1)(1)(3)(5) = -15.$

Answer:
-15

2. Use mathematical induction to prove that if n is any positive integer, then $1^3+2^3+\cdots+n^3 = (\frac{n(n+1)}{2})^2$.

Let $P(n)$ be the statement ``$1^3+2^3+\cdots+n^3 = (\frac{n(n+1)}{2})^2$.''

Base step:

Since $1^3 = 1$ and $(\frac{(1)(1+1)}{2})^2 = 1$, $P(1)$ holds.

Inductive step:

Suppose $k \geq 1$. Assume $P(k)$ holds, that is, assume that

$$1^3+2^3+\cdots+k^3 = (\frac{k(k+1)}{2})^2.$$

We must show that $P(k+1)$ holds, that is, we must show that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2.$$

Adding $(k+1)^3$ to both sides of the equation $P(k)$ gives

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (\tfrac{(k)(k+1)}{2})^2 + (k+1)^3 \\
&= (k+1)^2(\tfrac{k^2}{4} + (k+1)) \\
&= (k+1)^2(\tfrac{(k+2)^2}{4}) \\
&= (\tfrac{(k+1)(k+2)}{2})^2.
\end{aligned}$$

Therefore, $P(k+1)$ follows from $P(k)$.

3. Suppose that the function f is defined recursively by $f(1) = 3$, $f(2) = 4$, and for $n \geq 3$, $f(n) = \frac{(f(n-1)f(n-2))}{2}$. Find $f(6)$.

$$\begin{aligned}
f(3) &= \tfrac{f(2)f(1)}{2} = \tfrac{(4)(3)}{2} = 6. \\
f(4) &= \tfrac{f(3)f(2)}{2} = \tfrac{(6)(4)}{2} = 12. \\
f(5) &= \tfrac{f(4)f(3)}{2} = \tfrac{(12)(6)}{2} = 36. \\
f(6) &= \tfrac{f(5)f(4)}{2} = \tfrac{(36)(12)}{2} = 216.
\end{aligned}$$

Answer:
216

4. Find the base 8 representation of 413. [Your answer should have the form $(\_\_\_)_8$.]

   `The largest power of` 8 `which divides` 413 `is` $8^2 = 64$. $413 = (6)(64) + 29 = (6)(8^2) + 29$. `Since` $29 = (3)(8) + 5$, `we get`

$$413 = (6)(8^2) + (3)(8) + 5,$$

   `so the base` 8 `representation of` 413 `is` $(635)_8$.

   `Alternatively,` $413 = 8 \cdot 51 + 5$ `and` $51 = 8 \cdot 6 + 3$, `so` $413 = 8 \cdot (8 \cdot 6 + 3) + 5 = 6 \cdot 8^2 + 3 \cdot 8 + 5$. `Therefore,` $413 = (635)_8$.

Answer:
$(635)_8$

5. Use the Sieve of Erotosthenes and the list below to find all prime numbers smaller than 80.

   `The primes smaller than` $\sqrt{80}$ `are` 2, 3, 5, `and` 7. `We cross out the multiples of each of these numbers which are larger than they are. The numbers not crossed out are prime.`

$$\begin{array}{c}
2\ 3\ \not{4}\ 5\ \not{6}\ 7\ \not{8}\ \not{9}\ 1\not{0}\ 11\ 1\not{2}\ 13\ 1\not{4}\ 1\not{5}\ 1\not{6}\ 17\ 1\not{8} \\
19\ 2\not{0}\ 2\not{1}\ 2\not{2}\ 23\ 2\not{4}\ 2\not{5}\ 2\not{6}\ 2\not{7}\ 2\not{8}\ 29\ 3\not{0}\ 31\ 3\not{2}\ 33\ 3\not{4} \\
3\not{5}\ 3\not{6}\ 37\ 3\not{8}\ 3\not{9}\ 4\not{0}\ 41\ 4\not{2}\ 43\ 4\not{4}\ 4\not{5}\ 4\not{6}\ 47\ 4\not{8}\ 4\not{9}\ 5\not{0} \\
5\not{1}\ 5\not{2}\ 53\ 5\not{4}\ 5\not{5}\ 5\not{6}\ 5\not{7}\ 5\not{8}\ 59\ 6\not{0}\ 61\ 6\not{2}\ 6\not{3}\ 6\not{4}\ 6\not{5}\ 6\not{6} \\
67\ 6\not{8}\ 6\not{9}\ 7\not{0}\ 71\ 7\not{2}\ 73\ 7\not{4}\ 7\not{5}\ 7\not{6}\ 7\not{7}\ 7\not{8}\ 79\ 8\not{0}
\end{array}$$

Answer:
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$ $43, 47, 53, 59, 61, 67, 71, 73, 79$

6. Show that if a and b are positive integers such that $a^2 - b^2$ is prime, then $a = b + 1$.

   $a^2 - b^2 = (a+b)(a-b)$ and $a + b \geq 2$, so if $a^2 - b^2$ is prime, $a - b = \pm 1$. Since primes are positive, $a - b = 1$, so $a = b + 1$.

7. Find a sequence of 20 consecutive positive integers each of which is composite.

   We note that if $1 \leq k \leq n$, then $k|(n! + k)$. Therefore, none of the consecutive integers $21! + 2$, $21! + 3$, $\cdots$, $21! + 21$ is prime, and there are 20 of them.

   Answer:

   $21! + 2,\ 21! + 3,\ \cdots,\ 21! + 21$

8. (a) Use the Euclidean Algorithm to find $gcd(135, 96)$. Be certain that each step of the Euclidean Algorithm is clear–you will receive no credit for finding $gcd(135, 96)$ by other means.

$$\begin{aligned} 135 &= 1(96) + 39 \\ 96 &= 2(39) + 18 \\ 39 &= 2(18) + 3 \\ 18 &= 6(3) \end{aligned}$$

   Since the last non-zero remainder is 3, $gcd(135, 96) = 3$.

   Answer: 3

(b) Find integers m and n such that $135m + 96n = gcd(135, 96)$.

`Rewriting the equations in 8a gives`

$$\begin{aligned} 135 - 1(96) &= 39 \\ 96 - 2(39) &= 18 \\ 39 - 2(18) &= 3 \\ 18 &= 6(3) \end{aligned}$$

`Substituting, we get`

$$\begin{aligned} 3 &= 39 - 2(18) \\ &= 39 - 2[96 - 2(39)] \\ &= 5(39) - 2(96) \\ &= 5[135 - 1(96)] - 2(96) \\ &= 5(135) + (-7)(96) \end{aligned}$$

Answer: $m = 5$, $n = -7$

9. Find all positive integers n that satisfy the equation $gcd(12, n) = lcm(12, n)$. Explain your answer fully. (You will receive no credit for an answer which is not properly justified.)

$gcd(12, n)|12$ and $12|lcm(12, n)$ so $gcd(12, n) \leq 12 \leq lcm(12, n)$. Therefore, if $gcd(12, n) = lcm(12, n)$, then $12 = gcd(12, n)$. The same argument shows that if $gcd(12, n) = lcm(12, n)$, then $n = gcd(12, n)$. Therefore, if $gcd(12, n) = lcm(12, n)$, then $n = 12$. Since $n = 12$ obviously satisfies the equation, the only solution is $n = 12$.

Answer: 12