MATH 290, Section 001 — Name: Solutions
Fall, 2011
Exam 3 — Student ID number: ____________

1. Define a relation R on the set $\mathbb{N}$ of natural numbers by xRy if and only if $x|y^2$. Determine whether or not R is a partial order on $\mathbb{N}$ and explain your answer

Since, for example, $2|4^2$ and $4|2^2$, letting $x = 2$ and $y = 4$ gives xRy and yRx even though $x \neq y$. Therefore, R is not anti-symmetric, so it is not a partial order.

☐ R is a partial order. ☒ R is not a partial order

Explanation:

R is not anti-symmetric.

2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 - 3n$. Find $f(\{2, 4, 6\})$. Be sure to use correct notation.

$f(\{2,4,6\}) = \{f(2), f(4), f(6)\} = \{(2^2 - 6), (4^2 - 12), (6^2 - 18)\} = \{-2, 4, 18\}$.

Answer:

$\{-2, 4, 18\}$

3. Let $f: \mathbb{Z} \to \mathbb{R}$ be given by $f(x) = \sqrt{|x|}$. Find $f^{-1}((0, 2))$. ($(0, 2)$ is the interval $\{x \in \mathbb{R} : 0 < x < 2\}$.)

$f^{-1}((0,2)) = \{n \in \mathbb{Z} : f(n) \in (0,2)\} = \{n \in \mathbb{Z} : 0 < \sqrt{|n|} < 2\} = \{n \in \mathbb{Z} : 0 < |n| < 4\} = \{-3, -2, -1, 1, 2, 3\}$.

Answer:

$\{-3, -2, -1, 1, 2, 3\}$

4. Suppose X is a set and $A \subseteq X$. Prove that $\chi_{(X\setminus A)}(x) = \overline{1} + \chi_A(x)$ for all $x \in X$. (Here $\chi_S\colon X \to \mathbb{Z}_2$ is the characteristic function of S.)

Proof:

By the definition of the characteristic function, $\chi_A(x) = \overline{1} \Leftrightarrow x \in A$ and $\chi_{(X\setminus A)}(x) = \overline{1} \Leftrightarrow x \in X \setminus A$. If $x \in A$, $\chi_{(X\setminus A)}(x) = \overline{0} = \overline{1} + \overline{1} = \overline{1} + \chi_A(x)$. On the other hand, if $x \in X \setminus A$, then $\chi_{(X\setminus A)}(x) = \overline{1} = \overline{1} + \overline{0} = \overline{1} + \chi_A(x)$. Therefore, for any $x \in X$, $\chi_{(X\setminus A)}(x) = \overline{1} + \chi_A(x)$.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Find subsets A and B of $\mathbb{R}$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

There are many examples. For one, let $A = (-\infty, 0]$ and $B = [0, \infty)$. Then $A \cap B = \{0\}$ so $f(A \cap B) = \{0\}$, whereas $f(A) = f(B) = [0, \infty)$, so $f(A) \cap f(B) = [0, \infty)$.

Answer:
$A = (-\infty, 0]$, $B = [0, \infty)$

6. Let $f\colon \{1, 2, 3\} \to \mathbb{N}$ be the function $\{(1, 4), (2, 6), (3, 12)\}$ and let $g\colon \mathbb{N} \to \mathbb{R}$ be $g(x) = \frac{x}{2}$. Write the function $g \circ f\colon \{1, 2, 3\} \to \mathbb{R}$ as a set of ordered pairs.

$g \circ f(1) = g(f(1)) = g(4) = 2$. $g \circ f(2) = g(f(2)) = g(6) = 3$. $g \circ f(3) = g(f(3)) = g(12) = 6$. Therefore, $g \circ f = \{(1, 2), (2, 3), (3, 6)\}$.

$g \circ f = \{(1, 2), (2, 3), (3, 6)\}$

In problems 7 and 8 determine whether the given function is one-to-one or onto. Check all boxes which apply, and no other boxes. Give reasons for your answers.

7. $f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n + 3$.

   `f` is one-to-one because if $m \neq n$, then $2m \neq 2n$, so $2m + 3 \neq 2n+3$. f is not onto because $f(m)$ is odd for all m so, for example, $\nexists m$ such that $f(m) = 6$.

   ☑ One-to-one ☐ Onto

8. $f\colon [0, \infty) \to [1, \infty)$ given by $f(x) = x^2 + 1$.

   f is one-to-one because $f(x) = f(y) \Rightarrow x^2 + 1 = y^2 + 1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm|y| \Rightarrow x = y$ because x and y are both non-negative. f is onto because if $y \in [1, \infty)$, then $f(\sqrt{y-1}) = y$.

   ☑ One-to-one ☑ Onto

9. Determine if the function $f\colon \mathbb{Z}_6 \to \mathbb{Z}_6$ given by $f(x) = 2x$ is a bijection. If f is a bijection, find its inverse, and if it is not a bijection, explain why it is not.

Inverse or explanation
f is not a bijection because $f(\overline{0}) = \overline{0} = f(\overline{3})$ even though $\overline{0} \neq \overline{3}$.

10. Prove that if A is any set, then there is no surjection $f\colon A \to \mathcal{P}(A)$, that is, prove Cantor's Theorem. (Here $\mathcal{P}(A)$ is the power set of A.)

Proof:

Suppose that $f\colon A \to \mathcal{P}(A)$ were a surjection. Let $S = \{x \in A : x \notin f(x)\}$. Since f is onto, there is an $s \in A$ such that $f(s) = S$. Either $s \in S$ or $s \notin S$. If $s \in S$, then $s \notin f(s) = S$, a contradiction. If $s \notin S$, then $s \notin f(s)$, so $s \in S$, again a contradiction. Therefore, the onto function f cannot exist.

11. Use the definition of a countable set to prove that $\mathbb{N} \times \{0, 1\}$ is countable. (In other words, find a bijection between the given set and $\mathbb{N}$.)

Answer:

Define $f\colon \mathbb{N} \times \{0, 1\} \to \mathbb{N}$ by $f(m, n) = 2m - n$. (Another way of saying this is that $f(m, 0)$ is $2m$ and $f(m, 1) = 2m - 1$.) Then f is onto because if y is even, $y = f(\frac{y}{2}, 0)$ and if y is odd, $y = f(\frac{y+1}{2}, 1)$. f is one-to-one because if $2m - n = 2m' - n'$, then $n = n'$ because if one of these were 0 and the other were 1, then one of $2m - n$ and $2m' - n'$ would be even and the other would be odd. Therefore, $2m = 2m'$ so $m = m'$. This means that $(m, n) = (m', n')$.