MATH 290, Section 001 Name `Solutions`
Fall, 2011
Exam 2 Student ID number____________

1. (a) Use the Euclidean Algorithm to find $GCD(94, 18)$.

$$\begin{aligned} 94 &= 18(5) + 4. \\ 18 &= 4(4) + 2. \\ 4 &= 2(2). \end{aligned}$$

`Since the last non-zero remainder is 2,` $GCD(94, 18) = 2$.

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| $GCD(94, 18) =$ 2 |

(b) Find integers $a$ and $b$ such that $94a + 18b = GCD(94, 18)$

`Working backwards from the second equation in problem 1a, we get:`

$$\begin{aligned} 2 &= 18 - 4(4) \\ &= 18 - 4(94 - 18(5)) \\ &= (-4)(94) + (21)(18). \end{aligned}$$

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| $a = -4$ | $b = 21$ |

2. Recall that $\mathcal{P}(S)$ denotes the power set of the set $S$. Find a set $A$ such that $A \cap \mathcal{P}(A) \neq \emptyset$, or, if there is no such set, explain clearly and using correct grammar why no such set exists. Be sure to use correct notation.

`There are many such examples. Probably the simplest is` $A = \{\emptyset\}$. `Then` $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$, `so` $\emptyset \in A \cap \mathcal{P}(A)$. `Another example would be` $A = \{1, \{1\}\}$.

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| $A = \{\emptyset\}$ | Explanation if appropriate: |

3. Find a collection $\{I_n : n \in \mathbb{N}\}$ of non-empty intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$ (that is, $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$) and $\cap_{n=1}^{\infty} I_n = \emptyset$.

   One way to do this is to let $I_n = (n, \infty)$ for all $n \in \mathbb{N}$. Then any element of $\cap_{n=1}^{\infty} I_n$ would be a real number which is larger than every natural number, and there is no such real number. Another choice of the $I_n$'s that works is to let $I_n = (0, \frac{1}{n})$.

Answer:

$I_n = (n, \infty)$

4. Explain what, if anything, is wrong with the following proof by mathematical induction that for any natural number $n$, $n = n+2$. (Note: An explanation is not simply an assertion that the proof is incorrect because the conclusion is incorrect, or that some step is incorrect because the conclusion is incorrect.)

   Let $P(n)$ be the statement "$n = n + 2$."

   Base step: We must prove $P(1)$. If $n = 1$, $n = n+2 \Rightarrow n^0 = (n+2)^0 \Rightarrow 1 = 1$, which is true. Therefore, $P(1)$ holds.

   Inductive step: Assume that $n$ is a natural number and that $P(n)$ is true. We must prove $P(n+1)$, that is, we must show that $n + 1 = (n + 1) + 2$. By the inductive hypothesis, $n = n + 2$. Adding 1 to both sides of this equation gives $n + 1 = (n + 2) + 1$. Therefore, $P(n) \Rightarrow P(n+1)$, so $P(n)$ holds for all natural numbers $n$.

   Explanation:

   The problem is that the argument in the base step does not prove that $P(1)$ is true. The fact that $P(1)$ implies a true statement does not mean that $P(1)$ holds, because any statement--true or false--implies a true statement.

5. Use mathematical induction to prove that if $k$ is a positive integer, then $(1+i)^{4k} = (-4)^k$, where $i$ is the imaginary unit, that is, $i^2 = -1$. Be sure to state explicitly the statement to which you are applying mathematical induction.

Let $P(k)$ be the statement ''$(1+i)^{4k} = (-4)^k$.''

Base step:

$(1+i)^{4\cdot 1} = ((1+i)^2)^2 = (1+2i-1)^2 = (2i)^2 = 2i^2 = -4 = (-4)^1$, so $P(1)$ holds.

Inductive step:

Assume $P(k)$ holds, that is, $(1+i)^{4k} = (-4)^k$. We must prove $P(k+1)$. $(1+i)^{4(k+1)} = (1+i)^{4k}(1+i)^4 = (-4)^k(1+i)^4$(by the inductive hypothesis) $=$ $(-4)^k(-4)$(by the proof in the base step) $=$ $(-4)^{k+1}$. Therefore, $P(k+1)$ holds. By the Principle of Mathematical Induction, $P(k)$ holds for all $k \geq 1$.

6. Suppose that $a_0 = 0$, $a_1 = 3$, and $a_n = 5a_{n-1} - 4a_{n-2}$ for $n > 1$. Use mathematical induction to prove that $a_n = 4^n - 1$ for all non-negative integers $n$. Be sure to state explicitly the statement to which you are applying mathematical induction.

Let $P(n)$ be the statement ‘‘$a_n = 4^n - 1$.’’

Base step:

$a_0 = 0 = 4^0 - 1$, and $a_1 = 3 = 4^1 - 1$, so $P(0)$ and $P(1)$ are true.

Inductive step:

Assume that $n \geq 2$ and $P(k)$ holds for all $k < n$. Then, in particular, $a_{n-1} = 4^{n-1} - 1$ and $a_{n-2} = 4^{n-2} - 1$. Therefore, $a_n = 5a_{n-1} - 4a_{n-2} = 5(4^{n-1} - 1) - 4(4^{n-2} - 1) = 5(4^{n-1}) - 5 - 4^{n-1} + 4 = 4(4^{n-1}) - 1 = 4^n - 1$. Hence, $P(n)$ holds. By the Principle of Complete Induction, $P(n)$ holds for all $n \geq 0$.

7. Let $A = \{w, x, y\}$ and $B = \{y, z\}$. Find $(A \times B) \setminus (A \cap B)^2$. Be sure to use correct notation.

$A \times B = \{(w, y), (w, z), (x, y), (x, z), (y, y), (y, z)\}$ and $A \cap B = \{y\}$, so $(A \cap B)^2 = \{(y, y)\}$. Therefore, $(A \times B) \setminus (A \cap B)^2 = \{(w, y), (w, z), (x, y), (x, z), (y, z)\}$.

Answer:
$\{(w, y), (w, z), (x, y), (x, z), (y, z)\}$

8. Let $A = \{1, 2, 3, 4\}$. Find a relation $R$ on $A$ which contains $(1, 2)$, $(2, 3)$, and $(1, 3)$ such that $R$ is symmetric but not transitive.

In order that $R$ be symmetric, it must include $(2, 1)$, $(3, 2)$, and $(3, 1)$. Since $R$ is not to be transitive, there must be $(x, y) \in R$ and $(y, z) \in R$ such that $(x, z) \notin R$. Since $(1, 2) \in R$ and $(2, 1) \in R$, if we do not include $(2, 2)$ in $R$, $R$ will not be transitive. Therefore, we can take $R$ to be $\{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}$.

Answer:
$\{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}$

9. Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y$ is a rational number. Prove that $\sim$ is an equivalence relation on $\mathbb{R}$.

Proof:

$\sim$ is reflexive because if $x \in \mathbb{R}$, then $x - x = 0$, which is rational.
$\sim$ is symmetric because if $x \sim y$, then $x - y$ is rational, so $y - x = -(x - y)$ is also rational, that is, $y \sim x$.
$\sim$ is transitive because if $x \sim y$ and $y \sim z$, then $x-y$ and $y-z$ are rational, so $x-z = (x-y)+(y-z)$ is also rational, that is, $x \sim z$.

10. Suppose that $m$ and $n$ are positive integers such that $m \equiv n (mod\ 3)$. Does it follow that $2^m \equiv 2^n (mod\ 3)$? Explain your answer.

If we take $m = 1$ and $n = 4$, then $m \equiv n(mod\ 3)$. However, $2^1 = 2$, and $2^4 = 16$, but $16 - 2 = 14$, which is not divisible by 3. Therefore, we can take $m = 1$, $n = 4$. (There are other possibilities, as well.)

| Answer and explanation: |
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| The answer is "No." $m = 1$, $n = 4$ provide a counterexample. |

11. Let $R$ be the relation on $\mathbb{N}$ defined by $xRy$ if and only if $x$ divides $2y$. For each of the following properties, put a check (✓) next to each property satisfied by $R$ and put an "X" next to each property not satisfied by $R$. Give a brief explanation for each answer.

- ☑ Reflexive
  If $x \in \mathbb{N}$, then $x|2x$ so $xRx$.
- ☒ Symmetric
  $1|2 \cdot 3$ so $1R3$, but $3 \nmid 2 \cdot 1$, so $3 \not R 1$.
- ☒ Anti-symmetric
  $1|2 \cdot 2$ so $1R2$, and $2|2 \cdot 1$, so $2R1$, but $1 \neq 2$.
- ☒ Transitive
  $4|2 \cdot 2$ so $4R2$, and $2|2 \cdot 1$, so $2R1$, but $4 \nmid 2 \cdot 1$ so $4 \not R 1$.
- ☒ An equivalence relation on $\mathbb{N}$
  $R$ is not symmetric (or transitive).