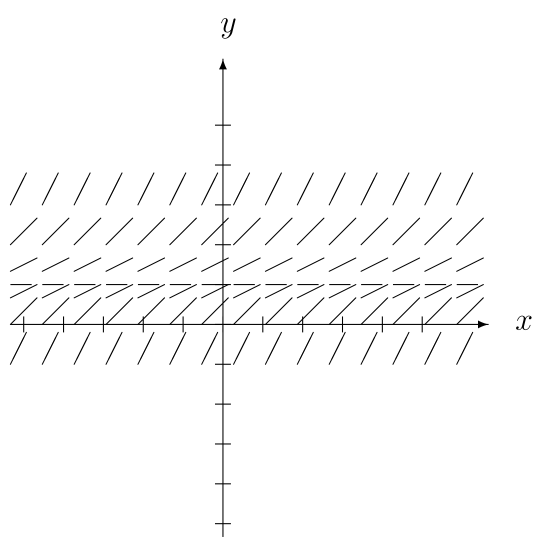
MATH 214, Section 001 Name ______ Solutions ______
Spring, 2011
Exam 1 Student ID number ____________

1. On the axes below, carefully draw the direction field for the differential equation $\frac{dy}{dx} = (y-1)^2$.

   Since $(y-1)^2 \geq 0$, all segments will have non-negative slope. Since $(y-1)^2 = 0$ if and only if $y = 1$, we draw horizontal segments only at points on the line $y = 1$. On the horizontal line $y = c$, where c is a constant, the slope of each segment will be $(c-1)^2$, so as c gets close to 1, the slope will get close to 0.



2. For each of the following differential equations, determine the given equation is linear or nonlinear (check one box), and determine the order of the differential equation.

(a) $(\frac{d^3y}{dx^3})^2 + x\frac{dy}{dx} = x$

- ☐ Linear
- ☑ Nonlinear

Order $= 3$

(b) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} + xy = 3x$

- ☑ Linear
- ☐ Nonlinear

Order $= 3$

(c) $\frac{d^3y}{dx^3} + y\frac{dy}{dx} + xy = 3x$

- ☐ Linear
- ☑ Nonlinear

Order $= 3$

In problems 3 through 7 solve the given ordinary differential equation or initial value problem

3. $\frac{dy}{dx} = (y-1)^2$, $y(1) = \frac{3}{2}$.

This is an autonomous ODE. Dividing by the right side gives $(y-1)^{-2}\frac{dy}{dx} = 1$. Integrating both sides of this equation with respect to x, and applying the Rule of Substitution gives $\int (y-1)^{-2}\frac{dy}{dx}dx = \int 1dx$, or $\int (y-1)^{-2}dy = \int 1dx$. Therefore, $-(y-1)^{-1} = x+C$, for a constant C. Solving gives $y = \left(\frac{-1}{x+C}\right)+1$. Since $y(1) = \frac{3}{2}$, we get $\frac{3}{2} = \left(\frac{-1}{1+C}\right) + 1$, so $\frac{-1}{1+C} = \frac{1}{2}$. Solving this equation for C gives $C = -3$. Therefore, a solution is given by $y = \left(\frac{-1}{x-3}\right) + 1$ $y = \left(\frac{1}{3-x}\right) + 1 = \frac{4-x}{3-x}$. (Note that this solution is valid for $x < 3$.)

Answer: $y = \frac{4-x}{3-x}$

4. $\frac{dy}{dx} + y = e^{-x}$

We multiply by the integrating factor $\mu(x) = e^{\int 1dx} = e^x$ to get $e^x \frac{dy}{dx} + e^x y = e^x e^{-x} = 1$. This equation can be written as

$$\frac{dye^x}{dx} = 1.$$

Integrating both sides with respect to x gives $ye^x = x + C$, or

$$y = xe^{-x} + Ce^{-x}.$$

Answer:
$y = xe^{-x} + Ce^{-x}$

5. $\frac{dx}{dt} + 2tx = 6t$, $x(0) = 8$.

Multiplying by the integrating factor $\mu(t) = e^{\int 2tdt} = e^{t^2}$ gives the equation

$$\frac{de^{t^2}x}{dt} = 6te^{t^2}.$$

Integrating both sides with respect to t gives $e^{t^2}x = \int 6te^{t^2}dt = 3e^{t^2} + C$, or $x(t) = 3 + Ce^{-t^2}$. Therefore, $x(0) = 3 + C$, and since we are given the initial condition $x(0) = 8$, $C + 3 = 8$, so $C = 5$. This gives the solution $x(t) = 3 + 5e^{-t^2}$.

Answer:
$x(t) = 3 + 5e^{-t^2}$

6. $\frac{dy}{dx} = \frac{2xe^y}{y}$

This equation is separable. It can be written as $ye^{-y}\frac{dy}{dx} = 2x$, or $ye^{-y}dy = 2xdx$. Integrating both sides of the equation gives $\int ye^{-y}dy = \int 2xdx$. The left integral can be evaluated using integration by parts:

$$\begin{array}{cc} u = y & v = -e^{-y} \\ du = dy & dv = e^{-y}dy \end{array}$$

Therefore, $\int ye^{-y}dy = \int udv = uv - \int vdu = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y}$. Since $\int 2xdx = x^2 - C$, we get the solution $-ye^{-y} - e^{-y} = x^2 - C$, or $ye^{-y} + e^{-y} = C - x^2$.

Answer:
$$ye^{-y} + e^{-y} = C - x^2$$

7. $(3x^2y + \sin y)dx + (x^3 + x\cos y - \sin y)dy = 0$.

$\frac{\partial(3x^2y+\sin y)}{\partial y} = 3x^2 + \cos y$ and $\frac{\partial(x^3+x\cos y-\sin y)}{\partial x} = 3x^2 + \cos y$. Therefore, the ODE is exact. We find a function $f(x,y)$ such that $f_x(x,y) = 3x^2y + \sin y$ and $f_y(x,y) = x^3 + x\cos y - \sin y$. Then $f(x,y) = \int(3x^2y + \sin y)dx = x^3y + x\sin y + \phi(y)$. This gives $f_y(x,y) = x^3 + x\cos y + \phi'(y) = x^3 + x\cos y - \sin y$. Therefore, $\phi'(y) = -\sin y$, so we can take $\phi(y) = \cos y$. Hence, one choice for $f(x,y)$ is $f(x,y) = x^3y + x\sin y + \cos y$. This gives the solution $x^3y + x\sin y + \cos y = C$.

Answer:
$$x^3y + x\sin y + \cos y = C$$

8. Find <u>two</u> solutions of the initial value problem $\frac{dy}{dx} = \left(\frac{3}{2}\right)\sqrt[3]{y}$, $y(1) = 0$. (Note that a solution must be defined on an open interval containing $x = 1$.)

One solution of the ODE is the constant $y = 0$, which satisfies the initial condition. To find another solution, write the ODE as $\left(\frac{2}{3}\right) y^{-\frac{1}{3}} \frac{dy}{dx} = 1$. Integrating gives $y^{\frac{2}{3}} = x + C$, or $y = \pm\left(\sqrt{x+C}\right)^3$. Since $y(1) = 0$, $\sqrt{1+C} = 0$, so $C = -1$. Therefore, $y = \pm\left(\sqrt{x-1}\right)^3$. However, these functions are not defined on an open interval containing $x = 1$. To get a second solution, we let $y = \begin{cases} 0 & \text{If } x \leq 0. \\ \left(\sqrt{x-1}\right)^3 & \text{If } x > 0. \end{cases}$

(We could let $y = \begin{cases} 0 & \text{If } x \leq 1. \\ -\left(\sqrt{x-1}\right)^3 & \text{If } x > 1. \end{cases}$)

Answer:

$$y = 0 \text{ and } y = \begin{cases} 0 & \text{If } x \leq 1. \\ \left(\sqrt{x-1}\right)^3 & \text{If } x > 1. \end{cases}$$