MATH 214, Section 001 Name `Solutions`
Spring, 2011
Exam 3 Student ID number________________

The following information might be helpful for some of the problems below.

- $x^3 - 7x^2 + 16x - 10 = (x-1)(x^2 - 6x + 10)$
- $(x-a)^4 = x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4$
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Some Laplace Transforms		
	$f(t)$	$\mathcal{L}(f(t))(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	e^{at}	$\frac{1}{s-a}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$\cos t + \frac{t^2}{2} - 1$	$\frac{1}{s^3(s^2+1)}$
7	$f^{(n)}(t)$	$s^n\mathcal{L}(f(t))(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

1. Find the general solution of the ODE $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 16\frac{dy}{dx} - 10y = 0$.

   The characteristic polynomial is $p(r) = r^3 - 7r^2 + 16r - 10 = (r-1)(r^2 - 6r + 10)$. From the quadratic formula, we get that the roots of $r^2 - 6r + 10$ are $r = 3 \pm i$. Therefore, the roots of $p(r)$ are 1 and $3 \pm i$, so the general solution of the ODE is $y = C_1e^x + C_2e^{3x}\cos x + C_3e^{3x}\sin x$.

Answer: $y = C_1e^x + C_2e^{3x}\cos x + C_3e^{3x}\sin x$

2. Find the general solution of the ODE $\frac{d^4y}{dx^4} - 12\frac{d^3y}{dx^3} + 54\frac{d^2y}{dx^2} - 108\frac{dy}{dx} + 81y = 0$.

   The characteristic polynomial is $p(r) = r^4 - 12r^3 + 54r^2 - 108r + 81 = (r-3)^4$. Therefore, $r = 3$ is a root of multiplicity 4, so the general solution of the ODE is $y = (C_1 + C_2x + C_3x^2 + C_4x^3)e^{3x}$.

Answer: $y = (C_1 + C_2x + C_3x^2 + C_4x^3)e^{3x}$

3. The general solution of the homogeneous ODE $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ is $y = C_1e^x + C_2e^{2x} + C_3e^{3x}$. Find the general solution of the non-homogeneous ODE $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 24x$.

   We look for a particular solution of the form $y = Ax + B$. Then
   $$y' = A,$$
   and
   $$y'' = y^{(3)} = 0.$$
   Substituting in the non-homogeneous ODE gives
   $$0 - 6(0) + 11A - 6(Ax + B) = 24x,$$
   or
   $$-6Ax + (11A - 6B) = 24x.$$

Comparing coefficients gives $-6A = 24$ and $11A - 6B = 0$. Therefore, $A = -4$ and $B = \frac{-22}{3}$. Therefore, a particular solution is $y = -4x - \frac{22}{3}$, so the general solution is $y = C_1e^x + C_2e^{2x} + C_3e^{3x} - 4x - \frac{22}{3}$.

Answer: $y = C_1e^x + C_2e^{2x} + C_3e^{3x} - 4x - \frac{22}{3}$

4. The general solution of the homogeneous ODE $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ is $y = C_1e^x + C_2e^{2x} + C_3e^{3x}$. Find the general solution of the non-homogeneous ODE $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x$.

Since $y = e^x$ is a solution of the homogeneous equation, we look for a particular solution of the form $y = Axe^e$. Then $y' = Ae^x + Axe^x$, $y'' = 2Ae^x + Axe^x$, and $y^{(3)} = 3Ae^x + Axe^x$. Substituting into the equation gives

$$(3Ae^x + Axe^x) - 6(2Ae^x + Axe^x) + 11(Ae^x + Axe^x) - 6Axe^x = e^x,$$

or

$$2Ae^x = e^x.$$

Therefore, $2A = 1$, so $A = \frac{1}{2}$. This gives the particular solution $y = \frac{xe^x}{2}$, so the general solution is $y = C_1e^x + C_2e^{2x} + C_3e^{3x} + \frac{xe^x}{2}$.

Answer: $C_1e^x + C_2e^{2x} + C_3e^{3x} + \frac{xe^x}{2}$

5. Find the Laplace transform $\mathcal{L}(t^2)$ of the function $f(t) = t^2$.

By definition, $\mathcal{L}(t^2)(s) = \int_0^\infty t^2e^{-st}dt$. We first use integration by parts to rewrite $\int t^2e^{-st}dt$. We let

$$\begin{array}{ll} u = t^2 & v = -\frac{e^{-st}}{s}, \\ du = 2tdt & dv = e^{-st}, \end{array}$$

we get $\int t^2 e^{-st}dt = -\frac{t^2e^{-st}}{s} + \int \frac{2te^{-st}}{s}dt$. Therefore, $\mathcal{L}(t^2)(s) = \int_0^\infty t^2 e^{-st}dt = \lim_{u\to\infty}(-\frac{t^2e^{-st}}{s}|_0^u) + \int_0^\infty \frac{2te^{-st}}{s}dt = (\frac{2}{s})\int_0^\infty te^{-st}dt = (\frac{2}{s})\mathcal{L}(t) = (\frac{2}{s})(\frac{1}{s^2}) = \frac{2}{s^3}$.

[An alternative argument is this: Let $f(t) = t^2$. Then $\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f'(0)$, that is, $\mathcal{L}(2t)(s) = s\mathcal{L}(t^2)(s) - f(0)$. Since $\mathcal{L}(2t)(s) = 2\mathcal{L}(t)(s) = 2(\frac{1}{s^2}) = \frac{2}{s^2}$ and $f(0) = 0$, $\frac{2}{s^2} = s\mathcal{L}(t^2)(s)$, so $\mathcal{L}(t^2)(s) = \frac{2}{s^3}$.]

Answer: $\mathcal{L}(t^2)(s) = \frac{2}{s^3}$

6. Suppose that $f(t)$ is a function such that $\frac{d^4f}{dt^4} = 7f(t)$, $f^{(3)}(0) = 0$, $f''(0) = 0$, $f'(0) = 0$, and $f(0) = 9$. Find $\mathcal{L}(f(t))(s)$.

$\mathcal{L}(f^{(4)}(t))(s) = s^4\mathcal{L}(f(t))(s) - s^3f(0) - s^2f'(0) - sf''(0) - f^{(3)}(0)$. From the initial conditions, we get $\mathcal{L}(f^{(4)}(t))(s) = s^4\mathcal{L}(f(t))(s) - 9s^3$. Since $f^{(4)}(t) = 7f(t)$, $\mathcal{L}(f^{(4)}(t))(s) = 7\mathcal{L}(f(t))(s)$, so $7\mathcal{L}(f(t))(s) = s^4\mathcal{L}(f(t))(s) - 9s^3$. Therefore, $(s^4-7)\mathcal{L}(f(t))(s) = 9s^3$, so $\mathcal{L}(f(t))(s) = \frac{9s^3}{s^4-7}$.

Answer: $\mathcal{L}(f(t))(s) = \frac{9s^3}{s^4-7}$

7. Suppose that $h(t)$ is a function such that $\mathcal{L}(h(t))(s) = \frac{1}{s^3}$. Solve the initial value problem $\frac{d^2x}{dt^2} + x = h(t)$, $x(0) = 2$, $x'(0) = 1$.

Taking Laplace transforms of both sides of the differential equation gives $\mathcal{L}(\frac{d^2x}{dt^2})(s) + \mathcal{L}(x(t))(s) = \mathcal{L}(h(t)(s)$. $\mathcal{L}(\frac{d^2x}{dt^2})(s) = s^2\mathcal{L}(x(t))(s) - sx(0) - x'(0) = \mathcal{L}(x(t))(s) - 2s - 1$. Therefore, $s^2\mathcal{L}(x(t))(s) - 2s - 1 + \mathcal{L}(x(t))(s) = \mathcal{L}(h(t))(s) = \frac{1}{s^3}$. Solving for $\mathcal{L}(x(t))(s)$ gives

$$\mathcal{L}(x(t))(s) = \frac{\frac{1}{s^3} + 2s + 1}{s^2+1} = \frac{1}{s^3(s^2+1)} + 2(\frac{s}{s^2+1}) + \frac{1}{s^2+1}.$$

From the table, we get

$$x(t) = \cos t + \frac{t^2}{2} - 1 + 2\cos t + \sin t = 3\cos t + \sin t + \frac{t^2}{2} - 1.$$

Answer:
$x(t) = 3\cos t + \sin t + \frac{t^2}{2} - 1$