MATH 214, Section 001 Name `Solutions`
Spring, 2011
Exam 2 Student ID number____________

1. Use Euler's method with step size $h = .1$ to estimate the value of $\phi(1.2)$ where $y = \phi(x)$ is a solution of the initial value problem $\frac{dy}{dx} = y^2 - x^2$, $y(1) = 2$.

   According to Euler's Method, to approximate a solution of $\frac{dy}{dx} = f(x, y)$. $y(x_0) = y_0$, we take $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + f(x_n, y_n)h$, where h is the step size. In this case, $h = .1$. Since $1.2 = 1 + 2h$, we want to find y_2. Applying the formula, we get $x_1 = 1.1$ and $y_1 = y_0 + f(x_0, y_0)h = 2 + (2^2 - 1^2)(.1) = 2.3$. Therefore, $y_2 = y_1 + f(x_1, y_1)h = 2.3 + (2.3^2 - 1.1^2)(.1) = 2.3 + .408 = 2.708$.

   $\boxed{\phi(1.2) \approx \quad 2.708}$

In problems 2 through 6 solve the given differential equation or initial value problem

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

   The characteristic equation for the differential equation is $r^2 + r - 6 = 0$, which can be written as $(r + 3)(r - 2) = 0$. Therefore, the roots of the characteristic polynomial are $r = -3$ and $r = 2$. This gives the general solution
   $y = C_1 e^{-3x} + C_2 e^{2x}$.

   $\boxed{\text{Answer: } y = C_1 e^{-3x} + C_2 e^{2x}}$

3. $\frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 29x = 0$

The characteristic polynomial is $r^2 - 10r + 29$. By the quadratic formula, the roots of this polynomial are given by

$$r = \frac{10 \pm \sqrt{100 - 116}}{2} = 5 \pm 2i.$$

Since the polynomial has complex roots, the solution is $x = e^{5t}(C_1 \cos(2t) + C_2 \sin(2t))$.

$$\boxed{\text{Answer: } x = e^{5t}(C_1 \cos(2t) + C_2 \sin(2t))}$$

4. $\frac{d^2y}{dx^2} + 9y = 0$, $y(\pi) = 4$, $y'(\pi) = 3$.

The characteristic polynomial is $r^2 + 9$, which has roots $\pm 3i$. Therefore, the general solution of the ODE is

$$y = C_1 \cos(3x) + C_2 \sin(3x).$$

Therefore,

$$y' = -3C_1 \sin(3x) + 3C_2 \cos(3x).$$

Since $y(\pi) = 4$, $C_1 \cos(3\pi) + C_2 \sin(3\pi) = 4$, that is, $-C_1 = 4$, so $C_1 = -4$. Since $y'(\pi) = 3$, $-3C_1 \sin(3\pi) + 3C_2 \cos(3\pi) = 0$, that is, $-3C_2 = 3$, so $C_2 = -1$. Therefore, the solution is $y = -4\cos(3x) - \sin(3x)$.

$$\boxed{\text{Answer: } y = -4\cos(3x) - \sin(3x)}$$

5. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$, $y(0) = 2$, $y'(0) = 1$.

The characteristic polynomial is $r^2 + 6r + 9 = (r+3)^2$. The only root of this polynomial is $r = -3$. Therefore, the general solution of the ODE is

$$y = C_1 e^{-3t} + C_2 t e^{-3t}.$$

The derivative is given by

$$y' = -3C_1 e^{-3t} + C_2 e^{-3t} - 3C_2 t e^{-3t}.$$

Since $y(0) = 2$, $C_1 = 2$. Since $y'(0) = 1$, $-3C_1 + C_2 = 1$; we know that $C_1 = 2$, so $C_2 = 7$. Therefore, the solution is $y = 2e^{-3t} + 7te^{-3t}$.

Answer:
$$y = 2e^{-3t} + 7te^{-3t}$$

6. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 52\cos(2x)$ (You may want to use an earlier problem.)

From problem 2 the general solution of the homogeneous ODE $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ is $y = C_1 e^{-3x} + C_2 e^{2x}$. We look for a particular solution of the given ODE of the form

$$y = A\cos(2x) + B\sin(2x).$$

Then

$$y' = -2A\sin(2x) + 2B\cos(2x),$$

and

$$y'' = -4A\cos(2x) - 4B\sin(2x).$$

We want y to be a solution of the given ODE, so we want $(-4A\cos(2x) - 4B\sin(2x)) + (-2A\sin(2x) + 2B\cos(2x)) - 6(A\cos(2x) + B\sin(2x)) = 52\cos(2x)$. Simplifying gives

$$(-10A + 2B)\cos(2x) + (-2A - 10B)\sin(2x) = 52\cos(2x).$$

Comparing coefficients gives

$$\begin{aligned} -10A + 2B &= 52. \\ -2A - 10B &= 0. \end{aligned}$$

Solving this system gives $A = -5$, $B = 1$. Therefore, a particular solution is given by $y = -5\cos(2x) + \sin(2x)$, so the general solution is

$$y = C_1 e^{-3x} + C_2 e^{2x} - 5\cos(2x) + \sin(2x).$$

Answer: $y = C_1 e^{-3x} + C_2 e^{2x} - 5\cos(2x) + \sin(2x)$

7. The function $y_1 = f(t) = t^5$ is a solution of the ODE $\frac{d^2y}{dt^2} - (\frac{3}{t})\frac{dy}{dt} - (\frac{5}{t^2})y = 0$ for $t > 0$. Use the method of reduction of order to find another solution y_2 of the ODE which, along with y_1, gives a fundamental set of solutions.

We look for a solution of the form

$$y_2 = t^5 h(t).$$

Then

$$y' = t^5 h'(t) + 5t^4 h(t),$$

and

$$y_2'' = t^5 h''(t) + 10t^4 h'(t) + 20t^3 h(t).$$

Substituting into the ODE gives $(t^5 h''(t) + 10t^4 h'(t) + 20t^3 h(t)) - (\frac{3}{t})(t^5 h'(t) + 5t^4 h(t)) - (\frac{5}{t^2})t^5 h(t) = 0$. This equation simplifies to $t^5 h''(t) + 7t^4 h'(t) = 0$, or

$$th''(t) + 7h'(t) = 0.$$

If we let $g(t) = h'(t)$, this equation becomes the separable first-order ODE $g'(t) + (\frac{7}{t})g(t) = 0$, or $\frac{g'(t)}{g(t)} = -\frac{7}{t}$. Integrating and simplifying gives $g(t) = \frac{1}{t^7}$. Therefore, $h'(t) = \frac{1}{t^7}$, so $h(t) = \frac{-1}{6t^6}$. Since the solution we are seeking is $y_2 = t^5 h(t)$, we get the solution $y_2 = \frac{-1}{6t}$. (Multiplying be a non-zero constant will still give a solution which forms a fundamental set of solutions with y_1, so we can also use the function $y_2 = \frac{1}{t}$.)

Answer:

$$y_2 = \frac{-1}{6t}$$