MATH 203, Section 001 Name `Solutions`
Fall, 2011
Exam 4

1. Let P_1 denote the vector space of polynomials of degree at most 1. Let $T\colon P_1 \to \mathbb{R}^3$ be given by $T(ax+b) = \begin{pmatrix} a \\ ab \\ a \end{pmatrix}$. Determine if T is a linear transformation. Give *clear* reasons for your answer.

$T(5(2x+3)) = T(10x+15) = \begin{pmatrix} 10 \\ 150 \\ 10 \end{pmatrix}$, `but` $5T(2x+3) = 5\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \\ 10 \end{pmatrix} \neq T(5(2x+3))$, `so` T `is not a linear transformation.`
`(Of course, there are many other examples that work.)`

☐ T is a linear transformation. ☒ T is not a linear transformation.

2. Suppose that $\mathcal{B}$ is the basis for $\mathbb{R}^2$ such that the change of coordinate matrix $P_\mathcal{B}$ from $\mathcal{B}$ to the standard basis is given by $P_\mathcal{B} = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$. If $[\vec{v}]_\mathcal{B} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\vec{v}$.

$\vec{v} = P_\mathcal{B}[\vec{v}]_\mathcal{B} = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Answer:
$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$

3. Suppose that $\mathcal{B}$ is the basis for $\mathbb{R}^2$ such that the change of coordinate matrix $P_{\mathcal{B}}$ from $\mathcal{B}$ to the standard basis is given by $P_{\mathcal{B}} = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$. Find the change of basis matrix from the standard basis to $\mathcal{B}$.

`The change of basis matrix from the standard basis to` $\mathcal{B}$ `is` $P_{\mathcal{B}}^{-1} = (\frac{1}{det(P_{\mathcal{B}})}) \begin{bmatrix} 0 & -1 \\ -3 & 4 \end{bmatrix} = (\frac{1}{-3}) \begin{bmatrix} 0 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & \frac{-4}{3} \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 0 & \frac{1}{3} \\ 1 & \frac{-4}{3} \end{bmatrix}$$

4. Let the bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{R}^2$ be given by $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

`The matrix` $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ `is given by` $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{C}} \right]$, `so we must solve the equations` $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ `and` $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. `We can do both of these simultaneously if we row-reduce the matrix whose first two columns are the elements of` $\mathcal{C}$ `and whose second two columns are the elements of` $\mathcal{B}$. `[Of course, it is fine to do these systems separately.]`

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

`Therefore,` $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

5. Suppose that $\mathcal{B}$ is the basis for $\mathbb{R}^3$ given by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $\mathcal{C}$ is a basis for $\mathbb{R}^3$ such that the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$. Find $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{C}}$.

`Since` $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. `Therefore,` $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

6. One eigenvalue of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is 2. Find an eigenvector corresponding to the eigenvalue 2.

`We want a non-zero vector` $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ `such that` $A\vec{v} = 2\vec{v}$`, that is, we want a non-trivial solution of the homogeneous system` $A\vec{v} - 2\vec{v} = \vec{0}$. `To find such a solution, we row-reduce the coefficient matrix.` $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$. `Therefore,` z `is free, with` $x = z$ `and` $y = \frac{z}{3}$. `Hence, one eigenvector is` $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

7. Find the characteristic polynomial $p(\lambda)$ of the matrix $A = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

$p(\lambda) = det(\lambda I - A) = det(\begin{bmatrix} \lambda & 2 & 0 \\ -2 & \lambda - 3 & 1 \\ 0 & -1 & \lambda - 1 \end{bmatrix}) = \lambda[(\lambda - 3)(\lambda - 1) + 1] + 2[2(\lambda - 1)] = \lambda^3 - 4\lambda^2 + 8\lambda - 4$.

$p(\lambda) =: \lambda^3 - 4\lambda^2 + 8\lambda - 4$

8. Let $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$. Find all eigenvalues of A.

The characteristic polynomial of A is $p(\lambda) = det(\lambda I - A) = det \begin{bmatrix} \lambda - 2 & 2 \\ 1 & \lambda - 4 \end{bmatrix} = \lambda^2 + 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$. Therefore, the eigenvalues are 2 and 5.

Answer:
2 and 5

9. Let $A = \begin{bmatrix} 3 & 5 & 0 & 1 \\ 2 & -1 & 2 & 1 \\ 1 & 2 & -1 & 0 \\ 6 & 3 & 0 & 2 \end{bmatrix}$. Then the characteristic polynomial of A is $p(\lambda) = \lambda^4 - 3\lambda^3 - 26\lambda^2 - 15\lambda + 7$. Find $A^5 - 3A^4 - 26A^3 - 15A^2 + A$. [Hint: The easiest way to do this is to use the Cayley-Hamilton Theorem.]

By the Cayley-Hamilton Theorem, $p(A) = A^4 - 3A^3 - 26A^2 - 15A + 7I = 0$.

Long division gives that $\lambda^5 - 3\lambda^4 - 26\lambda^3 - 15\lambda^2 + \lambda = \lambda p(\lambda) - 6\lambda$, so $A^5 - 3A^4 - 26A^3 - 15A^2 + A = AP(A) - 6A = A \cdot 0 - 6A = \begin{bmatrix} -18 & -30 & 0 & -6 \\ -12 & 6 & -12 & -6 \\ -6 & -12 & 6 & 0 \\ -36 & -18 & 0 & -12 \end{bmatrix}$.

Answer: $\begin{bmatrix} -18 & -30 & 0 & -6 \\ -12 & 6 & -12 & -6 \\ -6 & -12 & 6 & 0 \\ -36 & -18 & 0 & -12 \end{bmatrix}$

10. Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$. Then the eigenvalues of A are -1 and 5, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix D and find D.

We take P to be $P_{\mathcal{B}}$ where $\mathcal{B}$ is a basis consisting of eigenvectors of A. Since $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors, we can take $P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. The matrix D has the corresponding eigenvalues on the main diagonal, so $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

$P =$: $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$D =$: $\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$