MATH 203, Section 001 Name `Solutions`
Fall, 2011
Exam 3

1. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \end{pmatrix}$ and let $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. Determine whether or not $\vec{b}$ is in the column space of A. *Be sure that your work makes clear how you are arriving and your answer.*

We row reduce the augmented matrix $\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 4 & -2 \\ -1 & -1 & 3 \end{array}\right)$, to get

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 4 & -2 \\ -1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -2 & -4 \\ 0 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $\vec{b} = -5\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, so $\vec{b}$ is in the column space of A.

☒ $\vec{b}$ is in the column space of A. ☐ $\vec{b}$ is not in the column space of A

2. Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \end{pmatrix}$ and let $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Determine whether or not $\vec{b}$ is in the null space of A. *Be sure that your work makes clear how you are arriving and your answer.*

$A\vec{b} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \neq \vec{0}$. Therefore, $\vec{b}$ is not in the null spaced of A

☐ $\vec{b}$ is in the null space of A. ☒ $\vec{b}$ is not in the null space of A

For problems 3 and 4, let $A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 2 & 6 & 1 & 0 & 1 \\ 4 & 12 & 3 & -4 & 5 \end{bmatrix}$. Then the row-reduced echelon form of A is $\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3. Find a basis for the null space of A.

`If` $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ `is in the null space of` A`, then from the row-reduced echelon matrix, we get` $x_1 = -3x_2 - 2x_4 + x_5$ `and` $x_3 = 4x_4 - 3x_5$ `so` $\vec{x} = \begin{bmatrix} -3x_2 - 2x_4 + x_5 \\ x_2 \\ 4x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. `Therefore, a basis is given by`

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Answer:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Find a basis for the column space of A.

`Since the pivot columns of` A `are the first and third, a basis for the column space is formed by the first and third columns of` A. `Therefore, a basis for the column space is` $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Answer:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

5. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x+y \\ 3x+y \end{bmatrix}$. Determine whether or not T is invertible, and if it is, find T^{-1}.

`The standard matrix for` T `is` $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$. `Since` $(2)(1)-(3)(1) = -1 \neq 0$, A `is invertible and` $T^{-1} = T_{A^{-1}}$. $A^{-1} = \frac{1}{(2)(1)-(3)(1)} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$. `Therefore,` $T^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = (\begin{bmatrix} -x+y \\ 3x-2y \end{bmatrix})$.

Answer:

$$T^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = (\begin{bmatrix} -x+y \\ 3x-2y \end{bmatrix})$$

6. Let $A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 5 \\ 5 & 2 & 0 \end{bmatrix}$. Find $detA$.

`Expanding along the third row we get` $detA = (5)det\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - (2)det\begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix} + (0)det\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = (5)(16) - (2)(8) + 0 = 64$.

Answer:
64

7. Suppose that A is a 4×4 matrix such that $detA = 3$, and that B is obtained from A by interchanging the first and second rows. Find $det(2B)$.

`Since` B `is obtained from` A `by interchanging two rows,` $detB = -detA = -3$. `Since the matrix` $2B$ `is obtained from` B `by multiplying each of four rows by 2,` $det(2B) = 2^4 detB = 16(-3) = -48$.

$det(2B) =$ -48

8. Let $A = \begin{bmatrix} 3 & 8 & -2 & 12 \\ 5 & 2 & 0 & -2 \\ 1 & 0 & 8 & 3 \\ 6 & 6 & 0 & 5 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Use Cramer's Rule and the fact that $detA = 40$ to find x_1 where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution of the system of equations $A\vec{x} = \vec{b}$.

`By Cramer's Rule,` $x_1 = \dfrac{det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix}}{detA} = \dfrac{det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix}}{40}$.

$det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix} = det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & 11 \end{bmatrix} = 880$. `Therefore,` $x_1 = \frac{880}{40} = 22$.

$x_1 =$ 22

9. For each of the following subsets of $\mathbb{R}^3$, determine if the subset is a subspace of $\mathbb{R}^3$.

(a) $W = \{\begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} : x, y \in \mathbb{R}\}$.

Since for each vector in W, the first and third component are different, the zero vector is not in W so W is not a subspace of $\mathbb{R}^3$.

☐ W is a subspace of $\mathbb{R}^3$. ☒ W is not a subspace of $\mathbb{R}^3$.

(b) $U = \{\begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} : x \in \mathbb{R}\}$.

Since $\vec{0} = \begin{bmatrix} 0 \\ 2 \cdot 0 \\ 0 \end{bmatrix}$, $\vec{0} \in U$. If $\vec{x} = \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y \\ 2y \\ 0 \end{bmatrix}$ are elements of U, then $\vec{x} + \vec{y} = \begin{bmatrix} x+y \\ 2(x+y) \\ 0 \end{bmatrix} \in U$. If $\vec{x} = \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} \in U$ and c is a scalar, then $c\vec{x} = \begin{bmatrix} cx \\ 2(cx) \\ 0 \end{bmatrix} \in U$. Therefore, U is a subspace of $\mathbb{R}^3$.

☒ U is a subspace of $\mathbb{R}^3$. ☐ U is not a subspace of $\mathbb{R}^3$.

10. Let $M_{2\times3}$ denote the vector space of 2×3 matrices, and let H the subset of $M_{2\times3}$ consisting of the 2×3 matrices having at least one entry which is 0. Determine whether or not H is a subspace of $M_{2\times3}$ and give a reason for your answer.

It is not the case that the sum of elements of H is always an element of H. For example, if $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, then A and B are both elements of H but $A+B$ is not an element of H. Therefore, H is not a subspace of $M_{2\times3}$.

☐ H is a subspace of $\mathbb{R}^3$. ☒ H is not a subspace of $\mathbb{R}^3$.

Reason:

The sum of elements in H is not necessarily in H.