MATH 203, Section 001 Name `Solutions`
Fall, 2011
Exam 2

1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+2y+3z \\ x-7y-z \end{bmatrix}$. Find the standard matrix of T.

`The columns of the standard matrix are` $T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$, `and` $T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, `so the standard matrix is` $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -7 & -1 \end{bmatrix}$.

Answer:
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -7 & -1 \end{bmatrix}$$

2. Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. Find $T(\begin{bmatrix} 5 \\ 3 \end{bmatrix})$. [Hint: $\begin{bmatrix} 5 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.]

$$T(\begin{bmatrix} 5 \\ 3 \end{bmatrix}) = T(2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 2T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + 3T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 2\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 9 \end{bmatrix}.$$

$$T(\begin{bmatrix} 5 \\ 3 \end{bmatrix}) = \begin{bmatrix} 4 \\ 10 \\ 9 \end{bmatrix}$$

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation whose standard matrix is $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & -2 \end{bmatrix}$. Find $T(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix})$.

T `is given by` $T(\vec{x}) = A\vec{x}$. `Therefore,` $T(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}) = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 44 \\ -10 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 44 \\ -10 \end{bmatrix}$$

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find the standard matrix of T.
[Hint: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.]

`Since` $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) - T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$. `Therefore, the standard matrix of` T `is` $\begin{bmatrix} 2 & -1 \\ 3 & -3 \\ -1 & 0 \end{bmatrix}$.

Answer:

$$\begin{bmatrix} 2 & -1 \\ 3 & -3 \\ -1 & 0 \end{bmatrix}$$

Problems 5 and 6 refer to the following matrices. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -3 & 2 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 4 & 1 \\ -2 & 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 5 & 1 \end{bmatrix}$

5. (a) Find $A + 3B$, or, if $A + 3B$ is not defined, write "$A + 3B$ is not defined" in the answer box.

$$A + 3B = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -3 & 2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 4 & 1 \\ -2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 \\ -1 & -3 & -1 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} 5 & 11 & 3 \\ -1 & -3 & -1 \end{bmatrix}$$

(b) Find $A + 3C$, or, if $A + 3C$ is not defined, write "$A + 3C$ is not defined" in the answer box.

`Since` A `is a` $2{\times}3$ `matrix and` C `is a` $3{\times}2$ `matrix,` $A{+}3C$ `is not defined.`

Answer:

$A + 3C$ `is not defined.`

6. (a) Find AB, or, if AB is not defined, write "AB is not defined" in the answer box.

`Since` A `is a` 2×3 `matrix and` B `is a` 2×3 `matrix,` AB `is not defined.`

Answer:

AB `is not defined.`

(b) Find AC, or, if AC is not defined, write "AC is not defined" in the answer box.

$$AC = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 5 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2(1)+(-1)(2)+0(5) & 2(3)+(-1)(0)+0(1) \\ 5(1)+(-3)(2)+2(5) & 5(3)+(-3)(0)+2(1) \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 9 & 17 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} 0 & 6 \\ 9 & 17 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 3 & 0 & -5 \\ 2 & 2 & 4 \\ 1 & 6 & -3 \end{bmatrix}$. Find a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$.

`Let` $B = (\frac{1}{2})(A + A^T) = (\frac{1}{2}) \left(\begin{bmatrix} 3 & 0 & -5 \\ 2 & 2 & 4 \\ 1 & 6 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 6 \\ -5 & 4 & -3 \end{bmatrix} \right) =$

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 5 \\ -2 & 5 & -3 \end{bmatrix}$$

`and let` $C = (\frac{1}{2})(A - A^T) = (\frac{1}{2}) \left(\begin{bmatrix} 3 & 0 & -5 \\ 2 & 2 & 4 \\ 1 & 6 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 6 \\ -5 & 4 & -3 \end{bmatrix} \right) =$

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}.$$

$B = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 5 \\ -2 & 5 & -3 \end{bmatrix}$

$C = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$

8. Determine whether the matrix $A = \begin{bmatrix} 3 & 6 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible, and if it is, find A^{-1}.

We row-reduce the matrix $A|I$.

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{3} & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & -4 & -4 & -\frac{2}{3} & 1 & 0 \\ 0 & -1 & 0 & -\frac{1}{3} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{6} & -\frac{1}{4} & 0 \\ 0 & -1 & 0 & -\frac{1}{3} & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{6} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{4} & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{4} & 1 \end{array}\right].$$

Therefore, A is invertible and $A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & 2 \\ \frac{1}{3} & 0 & -1 \\ -\frac{1}{6} & -\frac{1}{4} & 1 \end{bmatrix}$.

Answer:

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & 2 \\ \frac{1}{3} & 0 & -1 \\ -\frac{1}{6} & -\frac{1}{4} & 1 \end{bmatrix}$$

9. Suppose that A is a 4×4 matrix such that $A^{-1} = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 5 & -2 \end{bmatrix}$.

Solve the system of equations $A\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 0 \end{bmatrix}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

`Multiplying both sides of the equation` $A\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 5 \\ 0 \end{bmatrix}$ `by` A^{-1} `gives`

$$\vec{x} = I\vec{x} = A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 4 \\ 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 5 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 13 \\ 16 \\ 27 \end{bmatrix}.$$

`Therefore,` $x_1 = 17$, $x_2 = 13$, $x_3 = 16$, `and` $x_4 = 27$.

$x_1 =$ 17	$x_2 =$ 13	$x_3 =$ 16	$x_4 =$ 27

10. Suppose that A is a 3×3 matrix such that $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, and

$B = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$. Find $(AB^{-1})^{-1}$.

$(AB^{-1})^{-1} = (B^{-1})^{-1}A^{-1} = BA^{-1} = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} =$
$\begin{bmatrix} 11 & -3 & 5 \\ 10 & 0 & 10 \\ -3 & 1 & 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} 11 & -3 & 5 \\ 10 & 0 & 10 \\ -3 & 1 & 1 \end{bmatrix}$