MATH 203, Section 001 Name `Solutions`
Fall, 2011
Exam 1

1. Find the augmented matrix for the system

$$\begin{cases} 3x_1 + 2x_2 = 7. \\ x_1 - x_2 + 4x_3 = 5. \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 7 \\ 1 & -1 & 4 & 5 \end{array}\right]$$

2. Find a matrix A and a vector $\vec{b}$ such that the system

$$\begin{cases} x_1 - 5x_2 + x_3 = 1. \\ 2x_1 + 3x_2 + x_3 = 2. \end{cases}$$

is the same as the equation $A\vec{x} = \vec{b}$.

$A = \begin{bmatrix} 1 & -5 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 2 & 4 & -8 \\ 0 & 4 & 12 \\ 1 & 2 & -4 \end{bmatrix}$. Find the row-reduced echelon form of A.

$$\begin{bmatrix} 2 & 4 & -8 \\ 0 & 4 & 12 \\ 1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 4 & 12 \\ 2 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 4 & 12 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} 1 & 0 & -10 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Use the augmented matrix to solve the following system of linear equations, or, if the system is inconsistent, write "Inconsistent" in the answer box.

$$\begin{cases} 3x_1 + 2x_2 = 4. \\ x_1 - x_3 = 1. \\ x_2 + x_3 = 0. \end{cases}$$

`The augmented matrix is` $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$. `Row-reducing we get`

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 3 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

`Therefore,` $x_1 = 2, x_2 = -1, x_3 = 1$.

Answer:

$x_1 = 2, x_2 = -1, x_3 = 1$

5. Determine whether or not the vector $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. If $\vec{w}$ is a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, show how to write $\vec{w}$ as a linear combination of the other vectors; otherwise, write in the answer box "$\vec{w}$ is not a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$."

We want to see if there are scalars x_1, x_2, and x_3 such that $x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3 = \vec{w}$. Therefore, we want to see if the following system has a solution: $\begin{cases} 2x_1 + x_3 = 5. \\ 2x_1 + x_2 + x_3 = 0. \\ x_1 + x_3 = 2. \end{cases}$ Row-reducing the augmented matrix gives $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 5 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right]$. Therefore, we can take $x_1 = 3$, $x_2 = -5$, and $x_3 = -1$, so $\vec{w} = 3\vec{v}_1 - 5\vec{v}_2 - \vec{v}_3$.

Answer:

$\vec{w} = 3\vec{v}_1 - 5\vec{v}_2 - \vec{v}_3$

6. Determine whether or not the vector $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$ is a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. If $\vec{w}$ is a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, show how to write $\vec{w}$ as a linear combination of the other vectors; otherwise, write in the answer box "$\vec{w}$ is not a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$."

We want to see if there are scalars x_1, x_2, and x_3 such that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{w}$. Therefore, we want to see if the following system has a solution: $\begin{cases} 2x_1 & & + & 2x_3 & = & 5. \\ 2x_1 & + & x_2 & & = & 0. \\ x_1 & & + & x_3 & = & 2. \end{cases}$ Row-reducing the augmented matrix gives $\left[\begin{array}{ccc|c} 2 & 0 & 2 & 5 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$. Since the last row corresponds to the equation $0 = 1$, the system has no solution, so $\vec{w}$ is not a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$..

Answer:
$\vec{w}$ is not a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

7. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 4 & 5 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ and let $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$. Find $A\vec{v}$ if it is defined, or, if it is not defined, write "$A\vec{v}$ is not defined" in the answer box.

$$A\vec{v} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 4 & 5 & 0 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} (1)(2) + (2)(2) + (-1)(6) \\ (0)(2) + (3)(2) + (-2)(6) \\ (4)(2) + (5)(2) + (0)(6) \\ (1)(2) + (3)(2) + (5)(6) \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 18 \\ 38 \end{bmatrix}.$$

Answer: $\begin{bmatrix} 0 \\ -6 \\ 18 \\ 38 \end{bmatrix}$

8. Determine whether or not the columns of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ span $\mathbb{R}^3$. Be sure that your work makes clear how you came up with your answer.

`We row-reduce the given matrix.`

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

`Since the row-reduced echelon form of the matrix has a row of 0s, the columns do not span` $\mathbb{R}^3$.

☐ The columns span $\mathbb{R}^3$.

☒ The columns do not span $\mathbb{R}^3$.

9. Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & -5 \end{bmatrix}$. Determine whether or not the homogeneous system $A\vec{x} = \vec{0}$ has a non-trivial solution. Be sure that your work makes clear how you came up with your answer.

`Row-reducing` A `we get` $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & -1 \\ 1 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -7 \\ 0 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{-7}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

`Since the row-reduced echelon form of A has at least one row containing at least two non-zero entries, the system has non-trivial solutions. Specifically, any vector of the form` $\begin{bmatrix} -2t \\ \frac{7t}{2} \\ t \end{bmatrix}$ `is a solution.`

☒ The system has non-trivial solutions.

☐ The only solution is the trivial solution.

10. Determine whether or not the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent. Be sure that your work makes clear how you came up with your answer.

`Let` A `be the matrix whose columns are the three given vectors, that is,` $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. `Row-reducing, we get`

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

`Since the homogeneous system` $A\vec{x} = \vec{0}$ `has only the trivial solution, the vectors are linearly independent.`

☒ The vectors are linearly independent.

☐ The vectors are not linearly independent.