MATH 114, Section 001
Spring, 2009
Exam 4

Name Solutions

1. Determine if the series $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges absolutely, converges conditionally, or diverges. Explain your answer.

We can use the Integral test to determine whether or not the series converges absolutely. Note that the function $f(x) = \frac{x}{x^2+1}$ is decreasing for $x > 1$: its derivative is $\frac{1-x^2}{(x^2+1)^2}$ which is less than 0 for $x > 1$. $\int_1^\infty \frac{x}{x^2+1}dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}dx = \lim_{t\to\infty} \frac{\ln(x^2+1)}{2}|_1^t$ $= \lim_{t\to\infty} \frac{\ln(t^2+1)}{2} - \frac{\ln(2)}{2} = \infty$. Therefore, the series does not converge absolutely .

On the other hand, since the absolute values of the terms of the series decrease and converge to 0, the Alternating Series Test implies that the series converges. Therefore, the series converges conditionally.

☐ Absolutely convergent ☒ Conditionally convergent ☐ Divergent

Explanation:

The series does not converge absolutely by the Integral Test but converges by the Alternating Series Test.

In problems 2 through 4, find the radius of convergence of the given series.

2. $\sum_{n=0}^{\infty} \frac{n(x-3)^n}{4^n}$

In general, the radius of convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$, where $c_n \neq 0$ for all n, is given by $\lim |\frac{c_n}{c_{n+1}}|$. In this case, we get that the radius of convergence is $\lim \frac{(\frac{n}{4^n})}{(\frac{n+1}{4^{n+1}})} = \lim \frac{(n \cdot 4^{n+1})}{(n+1)\cdot 4^n} =$ $\lim \frac{4n}{n+1} = 4$.

Answer: 4

3. $\sum\limits_{n=0}^{\infty} n^n(x-3)^n$

`The radius of convergence of the series` $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$`, where` $c_n \neq 0$ `for all` n`, is given by` $\lim|\frac{1}{\sqrt[n]{c_n}}|$`. In this case, we get that the radius of convergence is` $\lim \frac{1}{\sqrt[n]{n^n}} = \lim \frac{1}{n} = 0$.

Answer:	
	0

4. $\sum\limits_{n=0}^{\infty} \frac{3^n x^n}{n^n}$

`In general, the radius of convergence of the series` $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$`, where` $c_n \neq 0$ `for all` n`, is given by` $\lim \frac{1}{\sqrt[n]{|c_n|}}$`, so in this case, the radius of convergence is` $\lim \frac{1}{\sqrt[n]{|\frac{3^n}{n^n}|}} = \lim \frac{n}{3} = \infty$.

Answer:	
	∞

In problems 5 and 6, find the interval of convergence of the given series.

5. $\sum\limits_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

The radius of convergence is $\lim |\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}}}| = \lim \frac{\sqrt{n+1}}{\sqrt{n}} = 1$, so the interval of convergence has endpoints -1 and 1. We have to check convergence at the endpoints. If $x = -1$, then the series is the alternating series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. If $x = 1$, the series is $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges because it is a p-series with $p = \frac{1}{2} < 1$. Therefore, the interval of convergence is $[-1, 1)$.

Answer: $[-1, 1)$

6. $\sum\limits_{n=1}^{\infty} \frac{n^n(x-5)^n}{2^n}$

The radius of convergence of the series is $\lim |\frac{1}{\sqrt[n]{\frac{n^n}{2^n}}}| = \lim \frac{2}{n} = 0$. Therefore, the series converges only at $x = 5$, so the interval of convergence is $\{5\}$.

Answer: $\{5\}$

7. Find the sum of the series $\sum\limits_{n=1}^{\infty} \frac{n}{5^n}$. [Hint: Differentiate both sides of the equation $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$.]

`If` $|x| < 1$, `the sum of the geometric series` $\sum\limits_{n=0}^{\infty} x^n$ `is given by`

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

`Differentiating both sides of the equation gives`

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

`Multiplying by` x `gives`

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

`Since the first term is` 0, `this also gives`

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2},$$

`and letting` $x = \frac{1}{5}$ `gives`

$$\sum_{n=1}^{\infty} \frac{n}{5^n} = \frac{\frac{1}{5}}{(1-\frac{1}{5})^2} = \frac{5}{16}.$$

Answer:	$\frac{5}{16}$

8. Use the facts that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ to find the coefficient of x^4 in the MacLaurin series for $f(x) = e^x \cos x$.

`Since` $f(x) = e^x \cos x = (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots)(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)$, `we can find the MacLaurin series for` f `by taking the Cauchy product. The coefficient of` x^4 `in the Cauchy product is` $1\cdot(\frac{1}{4!})-(\frac{1}{2!})(\frac{1}{2!})+(\frac{1}{4!})\cdot 1 = \frac{1}{24}-\frac{1}{4}+\frac{1}{24} = -\frac{1}{6}$.

Answer: $-\frac{1}{6}$

9. Find the fifth degree Taylor polynomial for $y = f(x) = x + \sin x$ about $x = \pi$.

`We make a table to summarize the information we need.`

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$	$\frac{f^{(n)}(\pi)}{n!}$
0	$x+\sin x$	π	π
1	$1+\cos x$	0	0
2	$-\sin x$	0	0
3	$-\cos x$	1	$\frac{1}{3!}$
4	$\sin x$	0	0
5	$\cos x$	-1	$\frac{-1}{5!}$

`Therefore, the fifth degree Taylor polynomial is`

$$\pi + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}.$$

Answer: $\pi + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}$

10. Find the coefficient of x^5 in the power-series expansion of $\sqrt[3]{1+x}$. Do not bother doing any complicated arithmetic. [Hint: $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$.]

From the formula for the binomial series, we get that the coefficient of x^5 is $\binom{\frac{1}{3}}{5} = \frac{(\frac{1}{3})(\frac{-2}{3})(\frac{-5}{3})(\frac{-8}{3})(\frac{-11}{3})}{5!} = \frac{880}{3^5 5!}$

Answer: $\frac{880}{3^5 5!}$