MATH 114, Section 002 Name Solutions
Spring, 2009
Exam 2

In problems 1 through 5, evaluate the given integral.

1. $\int x \ln x dx$.

We use integration by parts. We let $u = \ln x$, so $dv = xdx$. We get the table

$u = \ln x$	$v = \frac{x^2}{2}$
$du = \frac{dx}{x}$	$dv = xdx$

Then $\int x \ln x dx = (\ln x)(\frac{x^2}{2}) - \int (\frac{x^2}{2}) \frac{dx}{x} = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

Answer:
$\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$

2. $\int \cos^3 x \sin^2 x dx$.

Writing $\cos^3 x$ as $\cos x(\cos^2 x) = \cos x(1 - \sin^2 x)$ we get $\int \cos^3 x \sin^2 x dx = \int \cos x(1 - \sin^2 x)(\sin^2 x) dx$. If we let $u = \sin x$, then $du = \cos x dx$, so $\int \cos^3 x \sin^2 x dx = \int \cos x(1 - \sin^2 x)(\sin^2 x) dx = \int (1 - u^2) u^2 du = \int u^2 - u^4 du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$.

Answer:
$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

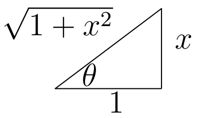
3. $\int \cos^2 x dx$

Using the identity $\cos^2 x = (\frac{1}{2})(1 + \cos(2x))$ and the formula $\int \cos(\alpha x) dx = (\frac{1}{\alpha}) \sin(\alpha x)$ gives $\int \cos^2 x dx = (\frac{1}{2}) \int (1 + \cos(2x)) dx = (\frac{1}{2})(x + \frac{\sin(2x)}{2}) + C = (\frac{1}{2})(x + \sin x \cos x) + C$. (The last equation uses the sine double-angle formula: $\sin 2x = 2 \sin x \cos x$.)

Answer:
$(\frac{1}{2})(x + \sin x \cos x) + C$

4. $\int \frac{1}{(x^2+1)^2}dx$. [Hint: An earlier problem might of use.]

We make the substitution $x = \tan\theta$, so $\theta = \arctan x$, $dx = \sec^2\theta d\theta$, and $x^2+1 = \sec^2\theta$. Substituting in the integral gives $\int \frac{1}{(x^2+1)^2}dx = \int \frac{\sec^2\theta}{(\sec^2\theta)^2}d\theta = \int \frac{1}{\sec^2\theta}d\theta = \int \cos^2\theta d\theta = (\frac{1}{2})(\theta + \sin\theta\cos\theta) + C$. (The last equality follows from problem 3.) To express the solution in terms of x we use the following diagram.



Since $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, we get that the integral is $(\frac{1}{2})(\arctan x + (\frac{x}{\sqrt{1+x^2}})(\frac{1}{\sqrt{1+x^2}})) + C = (\frac{1}{2})((\arctan x) + \frac{x}{1+x^2}) + C$.

Answer:

$$(\frac{1}{2})((\arctan x) + \frac{x}{1+x^2}) + C$$

5. $\int \frac{x^3}{\sqrt{9-x^2}}dx$

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta d\theta$, and $9 - x^2 = 9\cos^2\theta$, so $\sqrt{9-x^2} = 3\cos\theta$. Therefore, $\int \frac{x^3}{\sqrt{9-x^2}}dx = \int \frac{27\sin^3\theta}{3\cos\theta} \cdot 3\cos\theta d\theta = 27\int \sin^3\theta d\theta = 27\int \sin^2\theta \sin\theta d\theta = 27\int (1-\cos^2\theta)\sin\theta d\theta$. If we let $u = \cos\theta$, then $du = -\sin\theta d\theta$, so $\sin\theta d\theta = -du$. Substituting, we get $\int \frac{x^3}{\sqrt{9-x^2}}dx = 27\int (1-\cos^2\theta)\sin\theta d\theta = 27\int (1-u^2)\cdot -du = -27(u - \frac{u^3}{3}) + C = 9u^3 - 27u + C = 9\cos^3\theta - 27\cos\theta + C$. Since $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, $\int \frac{x^3}{\sqrt{9-x^2}}dx = (\frac{1}{3})(9-x^2)^{\frac{3}{2}} - 9\sqrt{9-x^2} + C$.

Answer:

$$(\frac{1}{3})(9-x^2)^{\frac{3}{2}} - 9\sqrt{9-x^2} + C$$

6. Find the partial fraction expansion of $\frac{6x^2-2x+6}{(x-1)(x^2+4)}$.

The form of the partial fraction expansion is

$$\frac{6x^2-2x+6}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}.$$

Multiplying by the denominator of the left side of the equation gives $6x^2-2x+6 = A(x^2+4)+(Bx+C)(x-1)$. Setting $x=1$ gives $5A = 10$, so $A = 2$. Therefore, $6x^2 - 2x + 6 = 2(x^2+4)+(Bx+C)(x-1) = (2+B)x^2+(C-B)x-C+8$. Comparing coefficients gives

$$\begin{aligned} 2+B &= 6. \\ -B+C &= -2. \\ -C+8 &= 6. \end{aligned}$$

From the first equation we get $B=4$, and from the last equation we get $C=2$. Therefore, $\frac{6x^2-2x+6}{(x-1)(x^2+4)} = \frac{2}{x-1} + \frac{4x+2}{x^2+4}$.

Answer:

$\frac{2}{x-1} + \frac{4x+2}{x^2+4}$

7. Estimate $\int_0^2 \sin(x^2)dx$ by dividing the interval $[0, 2]$ into six subintervals of equal length and applying the Trapezoidal Rule. Do not do any complicated arithmetic.

The length of each subinterval is $\Delta x = \frac{2}{6} = \frac{1}{3}$. The endpoints of the intervals are $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1, x_4 = \frac{4}{3}, x_5 = \frac{5}{3}$, and $x_6 = 2$. Therefore, the Trapezoidal Rule gives the approximation

$$\begin{aligned}\int_0^2 \sin x^2 dx &\approx (\frac{\Delta x}{2})(\sin x_0^2 + 2\sin x_1^2 + \\ &\qquad 2\sin x_2^2 + 2\sin x_3^2 + 2\sin x_4^2 + 2\sin x_5^2 + \sin x_6^2) \\ &= (\frac{1}{6})(\sin(0)^2 + 2\sin(\frac{1}{3})^2 + \\ &\qquad 2\sin(\frac{2}{3})^2 + 2\sin 1^2 + 2\sin(\frac{4}{3})^2 + 2\sin(\frac{5}{3})^2 + \sin 2^2) \\ &= (\frac{1}{6})(2\sin(\frac{1}{9}) + \ 2\sin(\frac{4}{9}) + 2\sin 1 + 2\sin(\frac{16}{9}) + 2\sin(\frac{25}{9}) + \sin 4).\end{aligned}$$

Answer:

$(\frac{1}{6})(2\sin(\frac{1}{9}) + \ 2\sin(\frac{4}{9}) + 2\sin 1 + 2\sin(\frac{16}{9}) + 2\sin(\frac{25}{9}) + \sin 4)$

In problems 8 through 10, determine whether or not the given improper integral converges, and if it does converge, evaluate it. If the improper integral diverges, write "Diverges" in the answer box.

8. $\int_1^\infty \frac{1}{x^6} dx$

$\int_1^\infty \frac{1}{x^6} = \lim_{t\to\infty} \int_1^t \frac{1}{x^6} dx = \lim_{t\to\infty} \int_1^t x^{-6} dx = \lim_{t\to\infty} \frac{-1}{5x^5}\Big|_{x=1}^t = \lim_{t\to\infty} \frac{-1}{5t^5} - (\frac{-1}{5}) = \frac{1}{5}$.

Answer:

$\frac{1}{5}$

9. $\int_0^\infty \frac{x}{x^2+1} dx$

$\int_0^\infty \frac{x}{x^2+1} dx = \lim_{t \to \infty} \int_0^t \frac{x}{x^2+1} dx = \lim_{t \to \infty} (\frac{1}{2}) \ln(x^2+1)|_{x=0}^t = \lim_{t \to \infty} (\frac{1}{2}) \ln(t^2+1) = \infty$. Therefore, the integral diverges.

Answer:
The integral diverges

10. $\int_{-1}^2 \frac{1}{x^3} dx$. (Careful: Read the instructions for problems 8 through 10.)

The improper integral will converge if and only if the improper integrals $\int_{-1}^0 \frac{1}{x^3} dx$ and $\int_0^2 \frac{1}{x^3} dx$ converge. $\int_{-1}^0 \frac{1}{x^3} dx = \lim_{t \to 0^-} \int_{-1}^t \frac{1}{x^3} dx = \lim_{t \to 0^-} \frac{-1}{2x^2}|_{x=-1}^t = \lim_{t \to 0^-} \frac{-1}{2t^2} + \frac{1}{2} = -\infty$. Since $\int_{-1}^0 \frac{1}{x^3} dx$ diverges, $\int_{-1}^2 \frac{1}{x^3} dx$ diverges as well.

Answer:
The integral diverges