MATH 114, Section 001 Name Solutions

Spring, 2009

Exam 1

1. A solid has for its base the unit disc $\{(x, y) : x^2 + y^2 \le 1\}$ in the xy plane. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

   If $A(x)$ represents the area of the cross section at x, then the volume is $\int_{-1}^{1} A(x)dx$. The cross section at x is a square with side $2\sqrt{1-x^2}$, the distance from the bottom of the disc to the top at x. Therefore, $A(x) = (2\sqrt{1-x^2})^2 = 4 - 4x^2$. Therefore, the volume is $\int_{-1}^{1}(4-4x^2)dx = (4x - \frac{4x^3}{3})|_{-1}^{1} = \frac{16}{3}$.

Answer:

$\frac{16}{3}$

For problems 2 and 3, let R the region below the graph of the function $y = f(x) = (2x - x^2)$, above the x-axis, for $0 \le x \le 2$.

2. Find the volume of the solid obtained by rotating R about the x-axis. Do not bother simplifying any messy arithmetic.

   Using the formula for rotating the region below the graph of $y = f(x)$ about the x-axis gives

$$
\begin{aligned}
V &= \pi \int_0^2 (2x - x^2)^2 dx \\
&= \pi \int_0^2 (4x^2 - 4x^3 + x^4) dx \\
&= \pi(\frac{4x^3}{3} - x^4 + \frac{x^5}{5})|_{x=0}^{2} \\
&= \pi(\frac{32}{3} - 16 + \frac{32}{5}) \\
&= \frac{16\pi}{15}.
\end{aligned}
$$

Answer:

$\frac{16\pi}{15}$

3. Find the volume of the solid obtained by rotating R about the y-axis. Do not bother simplifying any messy arithmetic.

`Using the formula for rotating the region below the graph of` $y = f(x)$ `about the` y`-axis gives`

$$\begin{aligned} V &= 2\pi \int_0^2 x(2x - x^2)dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)dx \\ &= 2\pi(\frac{2x^3}{3} - \frac{x^4}{4})|_{x=0}^2 \\ &= 2\pi(\frac{16}{3} - 4) \\ &= \frac{8\pi}{3}. \end{aligned}$$

Answer:
$\frac{8\pi}{3}$

4. Find the length of the graph of the function $y = f(x) = (\frac{2}{3})x^{\frac{3}{2}}$ for $0 \leq x \leq 8$. Do not bother simplifying any messy arithmetic.

`The length is given by` $\int_0^8 \sqrt{1 + (f'(x))^2}dx$. `Since` $f'(x) = x^{\frac{1}{2}}$, $(f'(x))^2 = x$. `Therefore, the length is` $\int_0^8 \sqrt{1 + x}dx$, `which we can evaluate using the Rule of Substitution: If` $u = 1 + x$, `then` $du = dx$, $u(0) = 1$, `and` $u(8) = 9$. `Therefore, the length is` $\int_0^8 \sqrt{1 + x}dx = \int_1^9 \sqrt{u}du = (\frac{2}{3})u^{\frac{3}{2}}|_{u=1}^9 = (\frac{2}{3})(27 - 1) = \frac{52}{3}$.

Answer:
$\frac{52}{3}$

5. In a particular colony the number $P(t)$ of bacteria at time t grows exponentially. At time $t = 0$, the population size is 1600. At time $t = 4$, the population size is 3600. Find a formula for the population $P(t)$ at time t.

   `Since` $P(t)$ `grows exponentially, there exist constants` C `and` r `such that` $P(t) = Ce^{rt}$. `Since` $P(0) = 1600$, $C = 1600$, `so` $P(t) = 1600e^{rt}$. $P(4) = 3600$ `and` $P(4) = 1600e^{4r}$ `so` $1600e^{4r} = 3600$. `Therefore,` $e^{4r} = \frac{3600}{1600} = \frac{9}{4}$, `or` $4r = \ln(\frac{9}{4})$, `so` $r = (\frac{1}{4})\ln(\frac{9}{4})$. `Substituting into the formula for` $P(t)$ `and simplifying gives` $P(t) = 1600e^{(\frac{1}{4})\ln(\frac{9}{4})t} = 1600(\frac{9}{4})^{\frac{t}{4}}$.

$P(t) =:$ $1600(\frac{9}{4})^{\frac{t}{4}}$

6. $1000 is invested in an account that compounds continuously. At what interest rate r will the amount in the account be $1500 at the end of 5 years? [You may use the fact that if $\$P$ is invested at interest rate r compounded continuously, then the amount in the account at the end of t years is $A(t) = Pe^{rt}$.]

   `In this case,` $A(t) = 1000e^{rt}$. `We want` r `so that` $A(5) = 1500$. `Therefore,` $1000e^{5r} = 1500$. `Dividing by` 1000 `gives` $e^{5r} = 1.5$ `or` $5r = \ln(1.5)$. `This gives` $r = \frac{\ln(1.5)}{5} = .2\ln(1.5)$.

$r =:$ $.2\ln(1.5)$

In problems 7 through 10, evaluate the given indefinite integral.

7. $\int x^2 \cos(x^3)dx$

We make the substitution $u = x^3$. Then $du = 3x^2dx$, so $x^2dx = \left(\frac{1}{3}\right)du$. Therefore, $\int x^2\cos(x^3)dx = \int \cos(x^3)x^2dx = \int \cos(u)\left(\frac{1}{3}\right)du = \left(\frac{1}{3}\right)\sin(u) + C = \left(\frac{1}{3}\right)\sin(x^3) + C$.

Answer:
$\left(\frac{1}{3}\right)\sin(x^3) + C$

8. $\int_0^4 \sqrt{4-(x-2)^2}dx$ [Hint: After making a substitution interpret the integral as an area.]

Let $u = x-2$. Then $du = dx$, $u(0) = -2$, and $u(4) = 2$. Therefore, $\int_0^4 \sqrt{4-(x-2)^2}dx = \int_{-2}^2 \sqrt{1-u^2}du$, which is the area of half of a circle of radius 2. Since the area of a circle of radius 2 is 4π, the integral is 2π.

Answer:
2π

9. $\int (\frac{x^2+2}{x+1})dx$

Using long division, we get that $\frac{x^2+2}{x+1} = x-1+\frac{3}{x+1}$. Therefore, $\int(\frac{x^2+2}{x+1})dx = \int x-1+\frac{3}{x+1}dx = \frac{x^2}{2} - x + 3\ln|x+1| + C$.

Answer:
$\frac{x^2}{2} - x + 3\ln\|x+1\| + C$

10. $\int \frac{1}{x^2+8x+17}dx$

Completing the square gives $x^2+8x+17=(x^2+8x+16)-16+17=(x+4)^2+1$. Therefore using the Rule of Substitution and the formulas for $\int(\frac{1}{u^2+1})du$, we get $\int \frac{1}{x^2+8x+17}dx=\int \frac{1}{(x+4)^2+1}dx=\arctan(x+4)+C$.

Answer:

$\arctan(x+4)+C$