MATH 112, Section 001 Name `Solutions`
Spring, 2009
Exam 4

1. In a group of 10 men and 12 women, 5 people are chosen at random to be on a committee. What is the probability that at exactly 2 men are chosen for the committee?

   The sample space S consists of the set of 5-element subsets of the 22 people, so $n(S) = C(22,5) = \frac{22!}{5!17!}$. S is an equiprobable space. Let E be the event that exactly 2 men are chosen. The number of ways that there can be exactly 2 men on the committee is the number of ways of choosing 2 of the 10 men and 3 of the 12 women, so $n(E) = C(10,2)C(12,3) = \left(\frac{10!}{2!8!}\right)\left(\frac{12!}{3!9!}\right)$. Therefore, $Pr(E) = \frac{n(E)}{n(S)} = \frac{C(10,2)C(12,3)}{C(22,5)} = \frac{5!17!10!12!}{22!2!3!8!8!} = \frac{50}{133}$.

   Answer: $\frac{C(10,2)C(12,3)}{C(22,5)}$ or $\frac{5!17!10!12!}{22!2!3!8!8!}$

2. Three fair dice are rolled. Find the probability that at least two come up the same.

   Let $S = \{(1,1,1),(1,1,2),\cdots,(6,6,6)\}$ be the sample space. Then S is equiprobable and $n(S) = 6^3 = 216$. Let E be the event that at least two of the dice are the same. Then E' is the event that all three are different. $n(E') = P(6,3) = 6{\cdot}5{\cdot}4 = 120$. Therefore, $Pr(E') = \frac{120}{216} = \frac{5}{9}$. It follows that $Pr(E) = 1 - Pr(E') = \frac{4}{9}$.

   Answer: $\frac{4}{9}$

3. At a particular university, 20% of the students smoke cigarettes, 50% of the students have missed at least three days of class because of illness, and 40% of the students who smoke have missed at least 3 days of class because of illness. If a student is chosen at random from the university, what is the probability that the person smokes and has missed at least 3 days of class because of illness? (Note: Not all of the given information is needed for the problem.)

Let E be the event that the person chosen smokes and let F be the event that the person chosen has missed at least 3 days of classes. We want $Pr(E \cap F)$. We are given that $Pr(E) = .2$ and $Pr((F|E) = .4$. Therefore, $Pr(E \cap F) = Pr(F|E)Pr(E) = (.2)(.4) = .08 = 8\%$. (The fact that 50% of the students have missed at least 3 days of class is not relevant to this problem.)

Answer:
8%

4. Three fair coins are tossed. Let E be the event that the first and last coins come up the same, and let F be the event that the first two coins come up heads. Are the events E and F independent? Explain.

Let the sample space be $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Then S is an equiprobable space and $n(S) = 8$. Then $E = \{HHH, HTH, THT, TTT\}$, so $Pr(E) = \frac{1}{2}$. Also, $F = \{HHH, HHT\}$, so $Pr(F) = \frac{1}{4}$. $E \cap F = \{HHH\}$ so $Pr(E \cap F) = \frac{1}{8} = Pr(E)Pr(F)$. Therefore, E and F are independent.

☒ E and F are independent.

☐ E and F are not independent.

Answer:
Since $Pr(E \cap F) = Pr(E)Pr(F)$, the events are independent.

5. A bag contains two coins. One is a usual fair coin. The other is a coin with both sides heads. A coin is drawn at random from the box and tossed. If it comes up heads, what is the probability that the other side is heads?

Denote the two heads on the bad coin as H_1 and H_2. Let the sample space be $S = \{(\text{The bad coin is drawn}, H_1),$ $(\text{The bad coin is drawn}, H_2),$ $(\text{The good coin is drawn}, H), (\text{The good coin is drawn}, H)\}$. If $E = \{(\text{The bad coin is drawn}, H_1), (\text{The bad coin is drawn}, H_2)\}$ and $F = \{(\text{The bad coin is drawn}, H_1), (\text{The bad coin is drawn}, H_2),$ $(\text{The good coin is drawn}, H)\}$, we want $Pr(E|F)$. Note that $E \cap F = \{(\text{The bad coin is drawn}, H_1), (\text{The bad coin is drawn}, H_2)\}$. Therefore, $Pr(E \cap F) = \frac{1}{2}$ and $Pr(F) = \frac{3}{4}$, so $Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)} = \frac{1/2}{3/4} = \frac{2}{3}$.

Answer:
$\frac{2}{3}$

6. Suppose that E_1, E_2, and E_3 form a partition of a sample space such that $Pr(E_1) = .1$, $Pr(E_2) = .2$, and $Pr(E_3) = .7$. Let F be an event such that $Pr(F|E_1) = .8$, $Pr(F|E_2) = .3$, and $Pr(F|E_3) = .6$. Find $Pr(E_1|F)$.

By Bayes' Theorem, $Pr(E_1|F) = \frac{Pr(F|E_1)Pr(E_1)}{Pr(F|E_1)Pr(E_1)+Pr(F|E_2)Pr(E_2)+Pr(F|E_3)Pr(E_3)} = \frac{(.8)(.1)}{(.8)(.1)+(.3)(.2)+(.6)(.7)} = \frac{.08}{.56} = \frac{1}{7}$.

Answer:
$\frac{1}{7}$

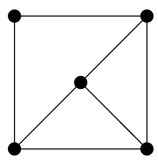
7. It is estimated that 10% of the population of a certain country has a particular disease. A test is developed for the disease. If the person has the disease, there is a 95% probability that the test comes back positive. If a person does not have the disease, there is a 1% probability that the test comes back positive. If a person tests positive, what is the probability that the person has the disease?

`Let` E `be the event that the person has the disease, and let` F `be the event that the person tests positive. We want` $Pr(E|F)$. `We will apply the special case of Bayes' Theorem. We are given that` $Pr(E) = .1$, `so` $Pr(E') = .9$. `We are also given that` $Pr(F|E) = .95$ `and` $Pr(F|E') = .01$. `Therefore, by Bayes' Theorem,` $Pr(E|F) = \frac{Pr(F|E)Pr(E)}{Pr(F|E)Pr(E)+Pr(F|E')Pr(E')} = \frac{(.95)(.1)}{(.95)(.1)+(.01)(.9)} = \frac{.095}{.104} = \frac{95}{104}$.

Answer:
$\frac{95}{104}$

8. Draw any simple graph having a total of 5 vertices such that one vertex has degree 2 and each of the other vertices has degree 3.

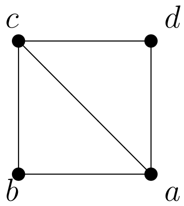
`One possibility is the following:`

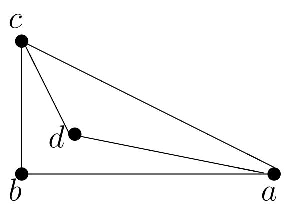


9. Determine if the graphs below are isomorphic. Explain your answer by either labelling the graphs in a way which indicates why there is an isomorphism or explaining why there is no such labelling.

☒ The graphs are isomorphic.

☐ The graphs are not isomorphic.





10. Determine if the graphs below are isomorphic. Explain your answer by either labelling the graphs in a way which indicates why there is an isomorphism or explaining why there is no such labelling.

☐ The graphs are isomorphic.

☒ The graphs are not isomorphic.

`The two graphs have different numbers of edges, so they are not isomorphic.`