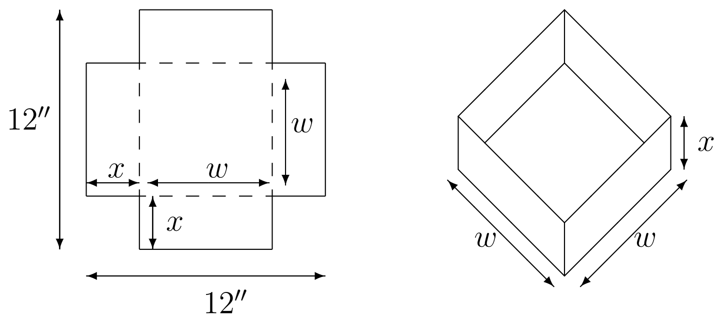
Math 108, Section A01 Name Solutions
Summer, 2009
Exam 4

Problems 1 and 2 refer to the following situation. A piece of 12 inch by 12 inch cardboard is to be made into a box without a top by cutting a square from each corner and folding up the resulting flaps. Let x denote the length of the sides of the squares that are removed. (Notice that $0 \leq x \leq 6$.)



1. Express the volume V of the box as a function of x.

   Let w denote the width of the box, which is also the length of the box. (See the picture.) Then since the height of the box is x, the volume of the box is $V = x \cdot w \cdot w = xw^2$. Since $w = 12 - 2x$, the volume is given by $V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3$.

   $V(x) = \ x(12-2x)^2$

2. What value of x gives the maximum volume of the box? Be sure to make clear why the value you find gives the maximum.

   We want the maximum of the function $V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3$, where $0 \leq x \leq 6$. Since $V'(x) = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(x-6)(x-2)$, the critical points of V are $x = 2$ and $x = 6$. Using the Extreme Value theorem, we evaluate V at $x = 0$, $x = 2$, and $x = 6$. $V(0) = 0$, $V(2) = (2)(12-4)^2 = 128$, and $V(6) = 0$. Since the largest value of V is $V(2) = 128$, the maximum volume is 128 cubic inches, so V is maximized when $x = 2$.

   Answer: 2

3. Solve the equation $5e^{2x} = 45$.

`Since` $5e^{2x} = 45$, $e^{2x} = 9$, `so` $2x = \ln 9$, `and` $x = (\frac{1}{2})\ln 9 = \ln(9^{\frac{1}{2}}) = \ln 3$.

Answer:
$\ln 3$

4. Suppose that \$1000 is deposited in an account that pays 8% interest . If interest is compounded continuously, how much is in the account at the end of six months (or one-half year)?

`For continuous compounding, the amount in the account at the end of` t `years is` Pe^{rt}, `where` P `is the initial deposit and` r `is the interest rate. In this case,` $P = 1000$, $r = .08$, `and` $t = \frac{1}{2}$ `so the amount in the account is` $1000e^{(.08)(\frac{1}{2})} = \$1000e^{.04}$.

Answer:
$\$1000e^{.04}$

5. The payroll of the Megagargantus Corporation has grown exponentially, that is, if $P(t)$ represents the payroll at time t years (where P is measured in millions of dollars), then $P(t) = Ce^{kt}$, where C and k are numbers. If $P(0) = 2$ and $P(3) = 16$, what is the payroll when $t = 1$, that is, what is $P(1)$?

$P(0) = Ce^0 = C$, `and since` $P(0) = 2$, $C = 2$. `Therefore,` $P(t) = 2e^{kt}$. `Since` $P(3) = 6$ `and` $P(3) = 2e^{3k}$, $2e^{3k} = 16$, `so` $e^{3k} = 8$ `or` $3k = \ln 8$. `Solving gives` $k = (\frac{1}{3})\ln 8 = \ln(8^{\frac{1}{3}}) = \ln 2$. `This gives` $P(t) = 2e^{(t\ln 2)} = 2(2^t)$, `so` $P(1) = 2(2) = 4$.

Answer:
4

In problems 6 through 10, find the derivative of the given function.

6. $f(x) = e^x + x^e + e^e$.

$f'(x) = \frac{d(e^x)}{dx} + \frac{d(x^e)}{dx} + \frac{d(e^e)}{dx} = e^x + ex^{e-1} + 0 = e^x + ex^{e-1}$.

Answer:
$e^x + ex^{e-1}$

7. $f(x) = xe^x + (\ln x)^2$

$f'(x) = \frac{d(xe^x)}{dx} + \frac{d(\ln x)^2}{dx} = [e^x \frac{d)}{dx} + x\frac{d(xe^x)}{dx}] + 2\ln x \frac{d \ln x}{dx} = e^x + xe^x + 2(\ln x)\frac{1}{x} = (1+x)e^x + \frac{\ln(x^2)}{x}$.

Answer:
$(1+x)e^x + \frac{\ln(x^2)}{x}$

8. $g(x) = 3e^{(x^3)} + \ln(x^2+1)$

$g'(x) = 3e^{(x^3)}\frac{dx^3}{dx} + \frac{1}{x^2+1}\frac{d(x^2+1)}{dx} = 3e^{(x^3)}(3x^2) + (\frac{1}{x^2+1})(2x) = 9x^2e^{(x^3)} + (\frac{2x}{x^2+1})$.

Answer:
$9x^2e^{(x^3)} + (\frac{2x}{x^2+1})$

9. $h(x) = \ln(\frac{\sqrt{x^2+3}}{\sqrt[3]{x^3+2}})$. [Hint: This problem is much easier if you first simplify using the properties of the logarithm.]

`We use the properties` $\ln\frac{a}{b} = \ln a - \ln b$ `and` $\ln(a^r) = r\ln a$ `of the logarithm. We get` $h(x) = \ln(\frac{(x^2+3)^{\frac{1}{2}}}{(x^3+2)^{\frac{1}{3}}}) = \frac{1}{2}\ln(x^2+3) - \frac{1}{3}\ln(x^3+2)$. `Therefore,` $h'(x) = (\frac{1}{2})(\frac{2x}{x^2+3}) - (\frac{1}{3})(\frac{3x^2}{x^3+2}) = \frac{x}{x^2+3} - \frac{x^2}{x^3+2}$.

Answer:
$\frac{x}{x^2+3} - \frac{x^2}{x^3+2}$

10. $f(x) = (x+1)^x$. [Hint: Use logarithmic differentiation.]

According to the rule of logarithmic differentiation, $f'(x) = f(x)\frac{d\ln f(x)}{dx}$, so $f'(x) = (x+1)^x[\frac{d\ln(x+1)^x)}{dx}] = (x+1)^x[\frac{dx\ln(x+1))}{dx}] = (x+1)^x[\ln(x+1)\frac{dx)}{dx} + x\frac{d\ln(x+1)}{dx}] = (x+1)^x[\ln(x+1) + \frac{x}{x+1}]$.

Answer: $(x+1)^x[\ln(x+1) + \frac{x}{x+1}]$