Math 108, Section A01 Name ______ Solutions ______
Summer, 2009
Exam 3

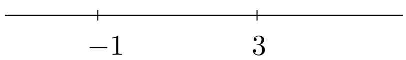
1. Find all critical numbers for the function $y = f(x) = x^3 + 6x^2 - 15x + 7$.

   We want the x-values such that $f'(x) = 0$. Since $f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x-1)(x+5)$, the critical numbers are $x = -5$ and $x = 1$.

Answer: -5, 1

2. Let $y = f(x) = \frac{1-x}{(x+1)^2}$. Find all intervals where f is increasing and all intervals where the graph of f is decreasing. You may use the fact that the derivative of f is given by $f'(x) = \frac{x-3}{(x+1)^3}$.

   The function $y = f(x)$ is not defined when $x = -1$ and $f'(x) = 0$ if $x = 3$, so $x = 3$ is a critical number.



   We can make the following table:

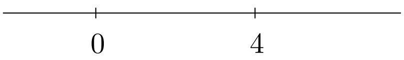
Interval	Test point	(Sign of) f'(Test point)
$(-\infty, -1)$	-2	$\frac{-5}{-27} > 0$
$(-1, 3)$	0	$-3 < 0$
$(3, \infty)$	4	$\frac{1}{5^3} > 0$

   Therefore, the function increases on $(-\infty, -1)$ and $(3, \infty)$ and decreases on $(-1, 3)$.

Increasing: $(-\infty, -1)$, $(3, \infty)$	Decreasing: $(-1, 3)$

3. Let $y = f(x) = x^4 - 8x^3 + 3x - 5$. Find all intervals the where the graph of f is concave up and all intervals where the graph of f is concave down.

$f'(x) = 4x^3 - 24x^2 + 3$, so $f''(x) = 12x^2 - 48x = 12x(x-4)$. Therefore, $f''(x) = 0$ if $x = 0$ and if $x = 4$.



We can make the following table:

Interval	Test point	(Sign of) f''(Test point)
$(-\infty, 0)$	-1	$12(-1)(-5) > 0$
$(0, 4)$	1	$12(1)(-3) < 0$
$(4, \infty)$	10	$12(10)(6) > 0$

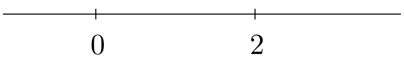
Therefore, the graph is concave up on the intervals $(-\infty, 0)$ and $(4, \infty)$ and concave down on the interval $(0, 4)$.

Concave up:
$(-\infty, 0)$, $(4, \infty)$

Concave down:
$(0, 4)$

4. Find all inflection points of the function $y = f(x) = 3x^5 - 20x^4 + 40x^3 - 7x - 7$. Do not put anything other than inflection points in the answer box. You may use the fact that $f''(x) = 60x(x-2)^2$. (Recall that an inflection point is a point where the concavity changes.)

Since $f''(x) = 60x(x-2)^2$, the only possible inflection points are $x = 0$ and $x = 2$.



We make a table to check concavity.

Interval	Test point	(Sign of) f''(Test point)
$(-\infty, 0)$	-1	$60(-1)(9) < 0$
$(0, 2)$	1	$60(1)(1) > 0$
$(2, \infty)$	3	$60(3)(1) > 0$

The graph is concave down on the interval $(-\infty, 0)$ and concave up on the intervals $(0, 2)$ and $(2, \infty)$. Since the concavity changes only at $x = 0$, the only inflection point is $(0, f(0)) = (0, -7)$.

Answer:
$(0, -7)$

5. Let $y = f(x) = (x-3)^3(x-8)^2$. Then $f'(x) = 5(x-3)^2(x-6)(x-8)$, so the critical numbers are 3, 6, and 8. Use the First Derivative Test to determine whether $x = 8$ gives a local minimum, a local maximum, or neither, and mark the correct answer. *If your work does not make clear that you are using the First Derivative Test correctly, you will receive no credit, even if your answer is correct!*

Using the given information, we can make a table. Since we are interested only in $x = 8$, we need only fill in the lines where the intervals include 8 as an endpoint.

Interval	Test point	(Sign of) f'(Test point)
$(-\infty, 3)$		
$(3, 6)$		
$(6, 8)$	7	$5(4^2)(1)(-2) < 0$
$(8, \infty)$	10	$5(7^2)(4)(2) > 0$

Since the derivative is negative to the left of $x = 8$ and positive to the right of $x = 8$, there is a local minimum at $x = 8$.

☒ Local minimum

☐ Local maximum

☐ Neither

6. Let $y = f(x) = x^3 - 6x^2 + 12x - 6$. Then $x = 2$ is a critical point of f. Use the Second Derivative Test to determine whether $x = 2$ gives a local minimum, a local maximum, or neither, or, if the Second Derivative Test gives no information, mark the box labeled 'Does not apply'. *Notice that this problem asks specifically what Second Derivative Test tells about the point where $x = 2$.*

$f'(x) = 3x^2 - 12x + 12$, and $f''(x) = 6x - 12$. Therefore, $f''(2) = 6(2) - 12 = 0$, so the Second Derivative Test does not give any information.

☐ Local minimum

☐ Local maximum

☐ Neither

☒ Does not apply

7. Let $y = f(x) = 2x^3 - 15x^2 + 36x + 1$. Then $x = 2$ is a critical point of f. Use the Second Derivative Test to determine whether $x = 2$ gives a local minimum, a local maximum, or neither, or, if the Second Derivative Test gives no information, mark the box labeled 'Does not apply'. *Notice that this problem asks specifically what Second Derivative Test tells about the point where $x = 2$.*

$f'(x) = 6x^2 - 30x + 36$, and $f''(x) = 12x - 30$. Therefore, $f''(2) = 24 - 30 = -6 < 0$. Since $f'(2) = 0$ and $f''(2) < 0$, $x = 2$ gives a local maximum.

☐ Local minimum

☒ Local maximum

☐ Neither

☐ Does not apply

8. Find all horizontal asymptotes of the function $y = f(x) = \frac{3x^5 - 2x^4 + x^2 - 5}{3x^4 - 6x^5}$. If there are no horizontal asymptotes, write "Does not exist" in the answer box.

The line $y = a$ is a horizontal asymptote of f if $\lim_{x\to\infty} f(x) = a$. Since the degree of the numerator and the degree of the denominator are the same, namely 5, the limit as x goes to ∞ of the given expression is the ratio of the coefficients of x^5 in the numerator and the denominator. Therefore, $\lim_{x\to\infty} \frac{3x^5 - 2x^4 + x^2 - 5}{3x^4 - 6x^5} = \frac{3}{-6} = -\frac{1}{2}$, so the line $y = -\frac{1}{2}$ is a horizontal asymptote.

Answer: $y = -\frac{1}{2}$

9. What is the smallest that a positive number plus the cube of its reciprocal can be, that is, what is the smallest value of $x + (\frac{1}{x})^3$ for x a positive number?

We want the minimum of the function $y = f(x) = x + (\frac{1}{x})^3 = x + \frac{1}{x^3} = x + x^{-3}$. $f'(x) = 1 - 3x^{-4} = 1 - \frac{3}{x^4} = \frac{x^4-3}{x^4}$. Therefore, the only positive value of x for which $f'(x) = 0$ is $x = \sqrt[4]{3}$. $f''(x) = 12x^{-5} = \frac{12}{x^5}$, so $f''(\sqrt[4]{3}) > 0$. Therefore, $x = \sqrt[4]{3}$ gives a local minimum, and since it is the only critical point, it gives the absolute minimum, so the minimum is $f(\sqrt[4]{3}) = \sqrt[4]{3} + (\frac{1}{\sqrt[4]{3}})^3$.

Answer: $\sqrt[4]{3} + (\frac{1}{\sqrt[4]{3}})^3$

10. Find the absolute minimum and the absolute maximum of the function $y = f(x) = x^3 - 9x^2$ on the interval $[1, 10]$. (Remember that the maximum and minimum are y-values, not x-values.)

$f'(x) = 3x^2 - 18x = 3x(x-6)$, so the critical numbers are $x = 0$ and $x = 6$. However, we are interested in the interval $[1, 10]$, and $x = 6$ is the only critical number in that interval. We evaluate f at the critical number and at each endpoint of the interval. $f(6) = 6^3 - 9(6^2) = -108$. $f(1) = 1^3 - 9(1^2) = -8$. $f(10) = 10^3 - 9(10^2) = 100$. Since the smallest y-value is -108 and the largest y-value is 100, the absolute minimum is -108 and the absolute maximum is 100.

Minimum: -108

Maximum: 100