

1. Find a number C such that the function f defined by $f(x) = \begin{cases} x^2 + 2 & \text{If } x \leq 3. \\ 2x + C & \text{If } x > 3. \end{cases}$ is continuous at $x = 3$. If no such number exists, write "Does not exist" in the answer box.

In order for f to be continuous at $x = 3$, it must be the case that $\lim_{x \rightarrow 3} f(x)$ exists and $\lim_{x \rightarrow 3} f(x) = f(3)$. In order for $\lim_{x \rightarrow 3} f(x)$ to exist, it must be the case that $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$. For $x < 3$, $f(x) = x^2 + 2$, so $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 + 2) = 3^2 + 2 = 11$. For $x > 3$, $f(x) = 2x + C$, so $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (2x + C) = 2(3) + C = 6 + C$. Therefore, $6 + C = 11$, so $C = 5$. If $C = 5$, $\lim_{x \rightarrow 3} f(x) = 11 = f(3)$.

Answer:

5

2. Use the limit definition of the derivative to find $f'(3)$ where $f(x) = \frac{2}{x}$. [No credit will be given unless the limit definition is used and your work is clear. Be sure to use limit notation correctly.]

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{2}{3+h}\right) - \left(\frac{2}{3}\right)}{h} = \lim_{h \rightarrow 0} \frac{\frac{6 - (6+2h)}{(3)(3+h)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{(3)(3+h)(h)} = \lim_{h \rightarrow 0} \frac{-2}{(3)(3+h)} = \frac{-2}{(3)(3+0)} = -\frac{2}{9}. \end{aligned}$$

Answer:

$-\frac{2}{9}$

In problems 3 through 7, find the derivative of the given function.

3. $f(x) = 3x^5 - 2x^3 + x - 5$.

$$f'(x) = 3 \frac{dx^5}{dx} - 2 \frac{dx^3}{dx} + \frac{dx}{dx} - \frac{d5}{dx} = 3(5x^4) - 2(3x^2) + 1 - 0 = 15x^4 - 6x^2 + 1.$$

Answer:

$15x^4 - 6x^2 + 1$

4. $g(x) = 6\sqrt[3]{x} + \frac{3}{x^2}$

$$g(x) = 6x^{\frac{1}{3}} + 3x^{-2}, \text{ so } g'(x) = 6\left(\frac{1}{3}x^{\left(\frac{1}{3}-1\right)}\right) + 3(-2x^{(-2-1)}) = 2x^{\left(-\frac{2}{3}\right)} - 6x^{-3} = \frac{2}{\sqrt[3]{x^2}} - \frac{6}{x^3}.$$

Answer:

$$\frac{2}{\sqrt[3]{x^2}} - \frac{6}{x^3}$$

5. $h(x) = \frac{x^2-2}{x^3+3}$

Using the Quotient Rule gives $h'(x) = \frac{(x^3+3)\frac{d(x^2-2)}{dx} - (x^2-2)\frac{d(x^3+3)}{dx}}{(x^3+3)^2} = \frac{(x^3+3)(2x) - (x^2-2)(3x^2)}{(x^3+3)^2} = \frac{6x+6x^2-x^4}{(x^3+3)^2}.$

Answer:

$$\frac{6x+6x^2-x^4}{(x^3+3)^2}$$

6. $f(x) = (x^4 - 3)^{50}$

Using the Power Chain Rule gives $f'(x) = 50(x^4 - 3)^{49}\frac{d(x^4-3)}{dx} = 50(x^4 - 3)^{49}(4x^3) = 200x^3(x^4 - 3)^{49}$

Answer:

$$200x^3(x^4 - 3)^{49}$$

7. $C(x) = (x^3 - 2)\sqrt{4x + 1}$

Write $C(x)$ as $(x^3 - 2)(4x + 1)^{\frac{1}{2}}$. Using the Product Rule and the Power Chain Rule gives $C'(x) = (4x+1)^{\frac{1}{2}}\frac{d(x^3-2)}{dx} + (x^3-2)\frac{d(4x+1)^{\frac{1}{2}}}{dx} = (4x+1)^{\frac{1}{2}}(3x^2) + (x^3-2)\left(\frac{1}{2}(4x+1)^{-\frac{1}{2}}(4)\right) = 3x^2\sqrt{4x+1} + \frac{2(x^3-2)}{\sqrt{4x+1}}.$
Using simple algebra, this can be simplified to $\frac{14x^3+3x^3-4}{\sqrt{4x+1}}.$

Answer:

$$\frac{14x^3+3x^3-4}{\sqrt{4x+1}}$$

8. Find the equation of the tangent line to the graph of $y = f(x) = \frac{9}{2x+1}$ at the point where $x = 1$.

The point $(1, f(1))$ is on the tangent line. Since $f(1) = \frac{9}{3} = 3$, the point $(1, 3)$ is on the tangent line. The slope of the tangent line is $f'(1)$. Using the Power Chain Rule, we get $f'(x) = \frac{d(9(2x+1)^{-1})}{dx} = (9)(-1)(2x+1)^{-2} \frac{d(2x+1)}{dx} = \frac{-18}{(2x+1)^2}$. Therefore, $f'(1) = \frac{-18}{(2+1)^2} = -2$, so the tangent line has equation $y - 3 = (-2)(x - 1)$, or $y = -2x + 5$.

Answer:

$$y - 3 = (-2)(x - 1)$$

9. Let $y = f(x) = x^3 - 3x^2$. Find all x -values where the tangent line to the graph of $y = f(x)$ is horizontal.

We want the x -values such that $f'(x) = 0$. $f'(x) = \frac{d(x^3 - 3x^2)}{dx} = 3x^2 - 6x$. Therefore, we must solve $3x^2 - 6x = 0$, or $x(x - 2) = 0$. Solving gives $x = 0$ and $x = 2$.

Answer:

$$x = 0, x = 2$$

10. Let $y = f(x) = x^4 - \frac{3}{x}$. Find the second derivative $f''(x)$.

$f(x) = x^4 - 3x^{-1}$, so $f'(x) = 4x^3 - 3(-1)x^{-2} = 4x^3 + 3x^{-2}$. Therefore, $f''(x) = (4)(3)x^2 + 3(-2)x^{-3} = 12x^2 - \frac{6}{x^3}$.

Answer:

$$12x^2 - \frac{6}{x^3}$$