Differential Topology Solution Set #1

Select Solutions

1. Prove that $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is an $n-$dimensional manifold.

   We make coordinate charts on S^n that cover the whole space. Let $U^+ = \{(x_1, \ldots, x_{n+1}) \in S^n$ such that $x_{n+1} > 0\}$. We define the diffeomorphism:

$$\phi : U^+ \longrightarrow V \subset \mathbb{R}^n$$

   by $\phi((x_1, \ldots, x_{n+1})) = (x_1, \ldots, x_n)$. Notice that ϕ is a one-to-one because $x_{n+1} = 1 - \sqrt{\sum_{i=1}^n (x_i)^2}$ (and in particular is determined by $x_1, \ldots, x_n$. ϕ is clearly infinitely differentiable, and its inverse is also clearly infinitely differentiable by the same methods introduced in the book for the case $n = 1$. Similarly, we define a different chart for the Southern hemisphere, where we use the same map on the set $U^- = \{(x_1, \ldots, x_{n+1}) \in S^n$ such that $x_{n+1} < 0\}$.

   We have now covered the sphere S^n with local charts except on the equator, a copy of S^{n-1} given by $\{(x_1, \ldots, x_n, 0) : \sum_i x_i^2 = 1\}$. Note that it would not be sufficient to use induction here, since we would not have shown that these points have neighborhoods in S^n that are locally diffeomorphic to $\mathbb{R}^n$.

   Instead, we note that the first paragraph could have been done for any hemisphere. For any i, let

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i > 0\}, \quad \text{and } U_i^- = \{(x_1, \ldots, x_{n+1}) \in S^n : x_i < 0\}$$

   and

$$\phi_i : U_i^+ \longrightarrow V \subset \mathbb{R}^n$$

   be given by $\phi_i((x_1, \ldots, x_{n+1})) = (x_1, \ldots, \hat{x}_i, \ldots x_{n+1})$, where $\hat{x}_i$ indicates omitting x_i. Then ϕ_i is clearly smooth. It is one-to-one (and therefore invertible) because $\phi_i^{-1}(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, 1 - \sqrt{\sum_{j=1}^n (y_i)^2}, y_{i+1} \ldots, y_n)$, where the term in the middle occurs in the i^{th} position. It is clear that ϕ_i^{-1} is (infinitely) differentiable – to check the ith term, one simply checks directly. So ϕ_i is a diffeomorphism. Similarly, $\phi_i : U_i^- \to V \subset \mathbb{R}^n$ defined the same way is a diffeomorphism. Clearly, any point on S^n is in U_i^+ or U_i^- for some i.

2. Section 1, #2

3. Section 1, #3

4. Section 1, #5 Show that every k-dimensional vector subspace V of $\mathbb{R}^N$ is a manifold diffeomorphic to $\mathbb{R}^k$, and that all linear maps on V are smooth.

   Choose any basis $\{v_1, \dots, v_k\}$ of V, where $v_i \in R^N$. Since V is k-dimensional, there are exactly k vectors in this basis, and the v_i are linearly independent. Choose a basis $\{e_1, \dots, e_k\}$ of $\mathbb{R}^k$. Let $\phi(v_i) := e_i$ for all i. Since $\{v_i\}$ are linearly independent, ϕ extends linearly to a map on all of V, and since the set $\{e_i\}$ is linearly independent, $\phi : V \to \mathbb{R}^k$ is surjective. It is clearly invertible for the same reasoning, where $\phi^{-1}(e_i) = v_i$, and $\phi^{-1} : \mathbb{R}^k \to V$ is the linear extension of the map on the basis elements. Now ϕ is one-to-one; the fact that it is smooth follows from the next exercise. Similarly, ϕ^{-1} is a linear map that is smooth by the next exercise. Thus ϕ is a diffeomorphism. The fact that all linear maps on V are smooth also follows from the next exercise.

5. Suppose that U is an open set in $\mathbb{R}^n$. Prove that if $L : U \to \mathbb{R}^m$ is a linear map, then $dL_x = L$ for all $x \in U$. *Hint.* Suppose $0 \in U$ and try to prove the statement first at $x = 0$.

   Let L be a linear map. Then for any $h \in \mathbb{R}^n$,

$$dL_x(h) = \lim_{t\to 0} \frac{L(x+th) - L(x)}{t} = \lim_{t\to 0} \frac{L(x) + tL(h) - L(x)}{t} = \lim_{t\to 0} \frac{tL(h)}{t} = L(h).$$

   It follows that $dL_x = L$ for any $x \in U$.

6. Section 2, #1

7. Section 2, #4

8. Section 2, #12 A *curve* in a manifold X is a smooth map $t \to c(t)$ of an interval of $\mathbb{R}^1$ into X. The *velocity vector* of the curve c at time t_o, denoted simply $\frac{dc}{dt}(t_0)$, is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0} : \mathbb{R}^1 \to T_{x_0}(X)$. In the case that $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \dots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \cdots, c_k'(t_0)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely.

The first step is straightforward: simply take the matrix $\left(\frac{dc_i}{dt}(t_0)\right)$ as is the definition of dc_{t_0}. Then we apply it to 1 by doing matrix multiplication:

$$\frac{dc}{dt}(t_0) = dc_{t_0}(1) = (\frac{dc_1}{dt}(t_0), \ldots, \frac{dc_k}{dt}(t_0)) \cdot 1 = \begin{pmatrix} \frac{dc_1}{dt}(t_0) \\ \cdot \\ \cdot \\ \frac{dc_k}{dt}(t_0) \end{pmatrix}$$

where this last expression is then equal to $(c_1'(t_0), \cdots, c_k'(t_0))$, as desired.

By construction, we have shown that the velocity vectors of curves in X that go through x are indeed elements of $T_x(X)$, which is the converse statement. To prove that every vector in $T_x(X)$ is a velocity vector of a curve, we have to solve the following differential equation (on a manifold — however, in this case the "manifold" is just $\mathbb{R}^k$!). Let $v \in T_x(X)$; we need to find a curve $c(t)$ such that $c(t_0) = x$ and $dc_{t_0}(1) = v$. In coordinates, we may write $x = (x_1, \ldots, x_k)$ and $v = (v_1, \ldots, v_k)$. Then we solve individually for the differential equation in one variable: $c_i(t_0) = x_i$, and $c_i'(t_0) = v_i$. By existence of solutions to differential equations, there is a solution to this equation (and initial condition). Let $c_i(t)$ be the solution. Then $c(t) = (c_1(t), \ldots, c_k(t))$ is the curve that has has velocity vector v.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the function

$$f(x, y) = (x^2 - y, x + y, 5).$$

Write out df in matrix form.

10. Prove that if $\phi : U \to V$ is a diffeomorphism of open sets in $\mathbb{R}^n$, then

$$(d\phi)^{-1} = d(\phi^{-1}).$$

Since ϕ is a diffeomorphism, it is invertible, and $\phi \circ \phi^{-1} = id$, the identity map. Then by the chain rule,

$$d(\phi \circ \phi^{-1}) = d\phi \circ d(\phi^{-1}) = d(id) = id.$$

Similarly, $d(\phi^{-1}) \circ d\phi = id$. On the other hand, by definition of inverses,

$$d\phi \circ (d\phi)^{-1} = (d\phi)^{-1} \circ d\phi = id.$$

The result follows by uniqueness of inverses.

Additional problems for graduate students, or undergraduate extra credit

11. Section 1, #10 "The" torus is the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy plane, where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$.

    To show that two spaces are diffeomorphic, one can give a (global) diffeomorphism. Let $T_{a,b}$ be the torus described; note that the circle of radius a in the xy plane is described by $C_a = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = a^2\}$ and $T_{a,b} = \{(x, y, z) : d((x, y, z), C_a) = b\}$ where d is the distance from any point (x, y, z) to the closest point on C_a. This "closest point" is obtained by dropping a perpendicular to the circle C_a. We begin by writing this in polar (NOT spherical) coordinates: let $C_a = \{(a, \theta, 0) : \theta \in [0, 2\pi)\}$. Then $T_{a,b} = \{(r, \theta, z) : (r - a)^2 + z^2 = b^2\}$. Meanwhile, we write $S^1 \times S^1$ in polar coordinates as well: $S^1 \times S^1 = \{(1, \eta_1, 1, \eta_2) : \eta_i \in [0, 2\pi)\} \subset \mathbb{R}^2 \oplus \mathbb{R}^2$.

    $$\phi : T_{a,b} \longrightarrow S^1 \times S^1$$

    be given by the following map:

    $$\phi((r, \theta, z)) = (1, \theta, 1, \eta),$$

    where η is the unique value in $[0, 2\pi)$ such that $b\cos(\eta) = r - a$ and $b\sin(\eta) = z$. (Note that I don't use the inverse trigonometric functions because they restrict their ranges in order to be well-defined functions). It is clear that the image of ϕ is indeed $S^1 \times S^1$, and that the map is one-to-one. The inverse map is

    $$\phi^{-1}(1, \eta_1, 1, \eta_2) = (r, \eta_1, z)$$

    where $r = b\cos(\eta_2) + a$ and $z = b\sin(\eta_2)$. It is clear that ϕ^{-1} is smooth, since cos and sin are smooth functions. To show that ϕ is smooth, one can take the derivative of ϕ at different points using the appropriate trigonometric function at the appropriate point. For example, $z > 0$, then use

    $$\phi_{z>0}((r, \theta, z)) = (1, \theta, 1, \arccos(\frac{r - a}{b}))$$

    is a diffeomorphism around these points. Under this (local) diffeomorphism, you always obtain that $\eta \in [0, \pi)$, so clearly $\phi_{z>0}$ is not a global diffeomorphism. However, this is the restriction of the global ϕ to the points where $z > 0$.

12. Section 1, #16

13. Section 2, #9(a-d)

14. Section 2, #10