Math 621: Algebra I
Mid-Term Exam
SOLUTIONS

1. Let G be a group and N a normal subgroup.

   (a) Show that G/N forms a group. Be sure to prove your operation is well-defined.

   Let gN and hN be two cosets in G/N. Define the operation by

   $$gNhN := (gh)N.$$

   To show it's well-defined, suppose that $g_1N = gN$. Then there exists $n \in N$ such that $g_1 = gn$. Then

   $$g_1NhN = (g_1h)N = (gnh)N.$$

   Since N is normal, there exists $n' \in N$ such that $nh = hn'$. Then

   $$(gnh)N = ghn'N = ghN,$$

   showing that $g_1NhN = gNhN$. Similarly, if $h'N = hN$, then $gNh_1N = gNhN$. Thus multiplication is well-defined.

   The element eN is clearly an identity for this multiplication, where $e \in G$ is G's identity. Then for any $gN \in G/N$, the element $g^{-1}N$ is its inverse, since $gNg^{-1}N = g^{-1}NgN = eN$. Lastly we show the operation is associative: $(gNhN)kN = ghNkN = (gh)kN = g(hk)N = gNhkN = gN(hNkN)$. The equality in the middle follows from the fact that the product on G is associative.

   (b) Prove that the two element subgroup $\{id, (12)\}$ is not normal in S_3. Note that normal implies $g(12)g^{-1} \in \{id, (12)\}$ for all $g \in S_3$. Let $g = g^{-1} = (13)$. Then $(13)(12)(13) = (23) \notin \{id, (12)\}$. Therefore, the subgroup is not normal.

2. Let G be a nontrivial group and $Z := Z(G)$ denote its center.

   (a) Prove that $Z \trianglelefteq G$.
   We show that $gZ = Zg$ for all $g \in G$. Note that $gZ = \{gz| \ z \in Z\}$. But $gz = zg$ since z is central. Then

$$gZ = \{gz| \ z \in Z\} = \{zg| \ z \in Z\} = Zg.$$

   (b) Suppose that the number of cosets $|G/Z|$ is prime. Prove that G is abelian.
   Since Z is normal in G, G/Z is a group. Then G/Z is cyclic (since it's prime order) and any nontrivial element is a generator. Let gZ be a generator. Then for any elements $a, b \in G$, there exist integers k_1, k_2 such that $aZ = (gZ)^{k_1} = g^{k_1}Z$ and $bZ = (gZ)^{k_2} = g^{k_2}Z$. It follows that there exists elements $z_1, z_2 \in Z$ such that $a = g^{k_1}z_1$ and $b = g^{k_2}z_2$. Lastly,

$$\begin{aligned} ab &= g^{k_1}z_1g^{k_2}z_2 \\ &= g^{k_1}g^{k_2}z_1z_2 \quad \text{since } z_1 \in Z. \\ &= g^{k_2}g^{k_1}z_2z_1 \quad \text{since } g \text{ commutes with itself,} \\ &\quad\quad \text{and elements of } Z \text{ commute with each other} \\ &= g^{k_2}z_2g^{k_1}z_1 \quad \text{since } z_2 \in Z. \\ &= ba \end{aligned}$$

   Thus G is abelian.

   (c) Let G be a nonabelian group of order 35. Prove that G has trivial center.
   G nonabelian implies $G \neq Z$. If Z is nontrivial, then by Lagrange, $|Z|$ has order 5 or 7. Since Z is normal, G/Z is a group, with order 7 or 5, respectively (again, by Lagrange). Since the quotient is prime, by part (2) the group G is abelian, contrary to assumption. Thus the center must be trivial.

3. (a) Find all conjugacy classes and their sizes in D_8.

Recall that $a \sim b$ if there exists $g \in G$ such that $b = gag^{-1}$. In this case we say that a and b are *conjugate*. A conjugacy class is a maximal set of elements, all of which are conjugate to each other. Most of you had the definition wrong.

In D_8, $\{1\}$ and $\{r^2\}$ are their own conjugacy classes, since these elements commute with everything. Note that NOTHING is conjugate to 1 except 1, since $g1g^{-1} = 1$ for all g. Similarly for r^2.

The other conjugacy classes are $\{r, r^3\}$, $\{s, sr^2\}$ and $\{sr, sr^3\}$. To see this, note that

$$s^l r^k (r)(s^l r^k)^{-1} = s^l r^k (r) r^{-k} s^{-l} = \begin{cases} r & \text{if } l = 0 \\ r^3 & \text{if } l = 1 \end{cases}$$

Similarly for the other conjugacy classes. Notice that conjugacy classes partition the group (unlike centralizers!).

(b) State the class equation and verify it for D_8. The class equation states that

$$|D_8| = |Z(D_8)| + \sum_{i=1}^{k} |G : C_G(g_i)|,$$

where the sum is over conjugacy classes, g_i is a representative for each class, and $C_G(g_i)$ is the centralizer of g_i in G. Since $|G : C_G(g_i)|$ is the size of the conjugacy class of g_i, we have that $8 = 2 + 2 + 2 + 2$, which is what we expect.

4. Recall that *Cauchy's Theorem* states that if G is a finite group, and p a prime dividing $|G|$, then G has an element of order p.

   (a) State the first part of Sylow's Theorem, concerning the existence of certain subgroups. Define any terms such as p-Sylow subgroups, that you use.

   Let G be a finite group with order $p^\alpha m$, where p does not divide m and $\alpha \geq 1$. Then there exists a subgroup of G of order p^α. (Such a subgroup is called a p-Sylow subgroup.)

   (b) Use Sylow's Theorem to prove Cauchy's Theorem.

   Let $|G| = p^\alpha m$. By Sylow's Theorem, G has a subgroup H of order p^α, with $\alpha \geq 1$. Choose any $x \in H$, $x \neq 1$. Then $\langle x \rangle$ is a cyclic subgroup of H, and by Lagrange, its order must divide the order of H. Since $|H| = p^\alpha$, $\langle x \rangle$ is a subgroup of order p^β for some $\beta \neq 0$. If $\beta = 1$, then x has order p and it is the element we seek. If $\beta > 1$, then $x^{p^{\beta-1}}$ has order p, since clearly $\left(x^{p^{\beta-1}}\right)^p = x^{p^\beta} = 1$, and if $\left(x^{p^{\beta-1}}\right)^k = 1$ for $k < p$, then x has order $p^{\beta-1}k$ in H, which is a contradiction because $p^{\beta-1}k$ doesn't divide p^β.