Math 621: Final Exam

Due by 4:30, Friday, May 4, 2007

You are allowed to use your text and your notes and previous problem sets. You are not allowed to use other notes, the internet, friends, colleagues, or professors in any way. You are not allowed to refer to other text books.

No credit will be given without proof! However, lots of partial credit will be given if you write down useful, relevant and correct steps towards a proof. Be sure to write neatly and coherently, and in complete sentences.

QUOTING THE BOOK: You are welcome to use any theorems in the book. However, you may not say ”by the result on page..” or “by example 4...” If you use any discussion in the text, you must re-explain it in your own words.

1. True or false: There exists a nonzero homomorphism of groups
$$\phi : \mathbb{Z}_{24} \longrightarrow \mathbb{Z}_{11} \oplus \mathbb{Z}_7.$$
Prove your answer.

2. Let G be a group of order 168 with no normal subgroups. How many elements of order 7 does G have?

3. True or false: As rings,
$$\mathbb{Q}[x]/\langle x^2 - 3\rangle \cong \mathbb{Q}[\sqrt{3}].$$
Prove your answer.

4. Recall that $M_n(R)$ is the ring of $n \times n$ matrices with entries in R, under the usual matrix operations of addition and multiplication. Let R be a ring and $I \subset R$ an ideal. Prove that
$$M_n(R)/M_n(I) = M_n(R/I).$$

5. Let G be a group of order 105 with a normal 7-Sylow subgroup and a normal 3-Sylow subgroup. Prove that G is abelian.

   Now suppose G has a normal 7-Sylow subgroup (but not necessarily a normal 3-Sylow subgroup), and there exists a 3-Sylow subgroup that centralizes the 7-Sylow subgroup. Is G still forced to be abelian? Prove your answer.

EXTRA CREDIT WILL BE GIVEN for anyone who can prove or disprove by counterexample the original statement: Let G be a group of order 105 with a normal 7-Sylow subgroup. Then G is abelian.

6. Prove that $x^{10} + x^9 + \cdots + x + 1$ is irreducible over $\mathbb{Z}$.