MATH 351
Solutions # 7

1. Suppose that X is a normal random variable with parameters $\mu = 1$ and $\sigma^2 = 9$.

   (a) Find $P\{-2 \le X \le 1\}$

   Solution. Since $X \sim N(1,9)$, we we have $(X-1)/3$ is standard normal, i.e. $Z = (X-1)/3 \sim N(0,1)$.

$$\begin{aligned} P(\{-2 \le X \le 1\}) &= P(\{(-2-1)/3 \le (X-1)/3 \le (1-1)/3\}) \\ &= P(\{-1 \le Z \le 0\}) = P(\{Z \le 0\}) - P(\{Z \le -1\}) \\ &= .5 - .1587 = .3413 \end{aligned}$$

   (b) Find $E[X]$ and $Var(X)$.

   Solution. The expected value and the variance are given by the parameters of the normal random variable. In particular, $E[X] = 1$ and $Var(X) = 9$.

   (c) What is the distribution of $Y = 2X - 1$? In other words, what kind of random variable is it, and what are its parameters?

   Solution. We find the probability density function of Y. We use the usual trick: First, we find its cumulative distribution and then take the derivative.

$$\begin{aligned} F_Y(t) &= P(\{Y \le t\}) = P(\{2X - 1 \le t\}) = P(X \le \frac{t+1}{2}\}) \\ &= \int_{\infty}^{\frac{t+1}{2}} \frac{1}{\sqrt{2\pi}\cdot 3} e^{-\frac{(x-1)^2}{2\cdot 9}}\, dx, \text{ since } X \sim N(1,9). \text{ Then,} \\ f_Y(t) = \frac{d}{dt}F_Y(t) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}\cdot 3} e^{-\frac{(\frac{t+1}{2}-1)^2}{2\cdot 9}} \\ &= \frac{1}{\sqrt{2\pi}\cdot 6} e^{-\frac{(t-4)^2}{2\cdot 36}} \text{ (by simplifying).} \end{aligned}$$

   This is the pdf for a normally distributed random variable with parameters $\mu = 4$ and $\sigma^2 = 36$. Therefore, $Y \sim N(4, 36)$.

2. Suppose that the heights of the men of a certain large city are normally distributed with mean 71 inches and variance 15 inches. Find the probability that a randomly chosen man is at least 5 feet tall and no more than 6'1" feet tall. What is the conditional probability that the randomly chosen man is at least 6 foot six, given that he is at least 6'1" feet tall?

   Solution. Let X be the height of a randomly chosen man. Then $X \sim N(71, 15)$, implying $\sigma = \sqrt{15}$. We note that $6'1''$ is 73 inches, and $6'6''$ is 78 inches.

$$\begin{aligned} P\{60 \leq X \leq 73\} &= P\{\frac{60-71}{\sqrt{15}} \leq \frac{X-71}{\sqrt{15}} \leq \frac{73-71}{\sqrt{15}}\} \\ &= P\{-2.850 \leq Z \leq .516\} = P\{Z \leq .516\} - P\{Z \leq -2.850\} \\ &= .6950 \end{aligned}$$

   Now for the conditional probability.

$$\begin{aligned} P\{X \geq 78 | X \geq 73\} &= \frac{P\{X \geq 78 \text{ and } X \geq 73\}}{P\{X \geq 73\}} = \frac{P\{X \geq 78\}}{P\{X \geq 73\}} \\ &= \frac{1 - P\{X \leq 78\}}{1 - P\{X \leq 73\}} = \frac{1 - P\{\frac{X-71}{\sqrt{15}} \leq \frac{78-71}{\sqrt{15}}\}}{1 - P\{\frac{X-71}{\sqrt{15}} \leq \frac{73-71}{\sqrt{15}}\}} = \frac{1 - P\{Z \leq \frac{7}{\sqrt{15}}\}}{1 - P\{Z \leq \frac{2}{\sqrt{15}}\}} \\ &= (1 - .9646)/(1 - .6972) = .1169. \end{aligned}$$

3. What is the probability that, of the first four men we meet from the city of problem 2, exactly four are at least 6'1" feet tall? What assumptions are you making?

   Solution. Assuming that all the heights are independent of one another, we can use the probabilities we have calculated. We find the probability that any particular man is at least 73 inches is $P\{X \geq 73\} = 1 - P\{X \leq 73\} = 1 - .6972 = .3028$. Then we use this probability to find the probability of picking four men who are all this height, letting Y be the standard binomial random variable with $n = 4$ and $p = .3028$. We obtain

$$P(\{Y = 4\}) = \binom{4}{4}(.3028)^4 = .0084.$$

4. Suppose that a fair coin is flipped 100 times. Find the probability that exactly 50 of the tosses are heads, using both the exact probability given by the binomial distribution and the approximation given by the normal distribution.

   Solution. The exact solution: Let $X \sim binom(100, .5)$ and compute

$$P(\{X = 50\}) = \binom{100}{50} .5^{50} .5^{50} = .0796.$$

   On the other hand we could approximate this using the DeMoivre Leplace Limit Theorem. Notice that $np = 50$ and $\sqrt{np(1-p)} = 5$, which says that,

$$\begin{aligned} P\{\frac{49.5-50}{5} \le \frac{X-50}{5} \le \frac{50.5-50}{5}\} &= P(\{-.1 \le Z \le .1\} \\ &\to \Phi(.1) - \Phi(-.1) = .5398 - .4602 = .0797. \end{aligned}$$

5. Suppose that 15% of the student population at small colleges consists of Gary Johnson fans (Libertarian candidate for president in most states). If a college has 300 students, what is the probability that at least 35 of them are Gary Johnson fans? What assumptions are you making?

   Solution. We assume that the college students support their candidates independent of each other. We also assume that, if X is the number of people on this campus supporting Johnson, then $X \sim binom(300, .15)$. We want to find $P(\{X \ge 35\})$. We use a normal approximation. Note that $np = 45$, and $\sqrt{np(1-p)} = \sqrt{45(.65)} = 5.4$ using the DeMoivre Leiplace theorem, we have

$$\begin{aligned} P\{X \ge 35\} = 1 - P\{X \le 35\} &= 1 - P\{\frac{X-45}{5.4} \le \frac{35-45}{5.4}\} \\ &= 1 - P(\{Z \le 1.852\} = 1 - .9680 = .0320 \end{aligned}$$

6. Suppose that X is exponentially distributed with $\lambda = 3$.

   (a) What is $P\{X > 2\}$?

   Solution.

$$P\{2 < X < \infty\} = \int_2^\infty 3e^{-3x}\,dx = e^{-6}.$$

(b) What is $P\{X > 5|X > 3\}$?

Solution.

$$P\{X > 5|X > 3\} = \frac{P\{X > 5 \text{ and } X > 3\}}{P\{X > 3\}} = \frac{P\{X > 5\}}{P\{X > 3\}}$$
$$= \left(\int_5^\infty 3e^{-3x}\, dx\right) / \left(\int_3^\infty 3e^{-3x}\, dx\right)$$
$$= \frac{e^{-15}}{e^{-9}} = e^{-6}$$

What did you notice about these two answers? Is it a coincidence?

They are the same, due to the memoryless property. Not a coincidence for exponential random variables!

7. Suppose X is exponentially distributed with $\lambda = 3$. Find a number m for which $P\{X \leq m\} = 0.5$. Find $\mu = E[X]$. Are μ and m the same?

   Solution. If $m \geq 0$ (which we may assume, since we're looking for positive probability), then

   $$P\{X \leq m\} = \int_0^m 3e^{-3x}\, dx = -e^{-3m} + 1 = .5, \text{ or } e^{-3m} = .5$$

   implies that $-3m = \ln(.5) = -.693$ and thus $m = .231$. (By the way, you can check you have the right answer in Excel by entering =EXPONDIST(.231,3,TRUE). It returns .500 (up to 3 decimals).

   Now we find the expected value for an exponential random variable is $1/\lambda = 1/3$, so m and μ are definitely not the same. This contrasts with the normal distribution, in which the value of m such that $P(X \leq m) = .5$ is the same as the expected value.

8. Evaluate the integral
   $$\int_0^\infty x^7 e^{-2x} dx.$$

   Solution. Save yourself some time and use the Gamma function! Recall that
   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

The integral we are asked to evaluate has a $-2x$ in the exponent of the exponential so we first use u-substitution with $u = 2x$.

$$\int_0^\infty x^7 e^{-2x}\,dx = \frac{1}{2^7}\int_0^\infty u^7 e^{-u} \cdot \frac{1}{2}\,du = \frac{1}{2^8}\Gamma(8) = \frac{7!}{2^8}.$$