MATH 351
Solutions #5

1. Let X be a Bernoulli random variable with parameter $p = \frac{5}{6}$. Find $E[cos(\pi X)]$, $E[3^X]$, and $E[\tan^{-1}(X)]$.

   Solution.

$$E[cos(\pi X)] = \sum_{x:p(x)>0} g(x)p(x) = \cos(\pi \cdot 0)(\frac{1}{6}) + \cos(\pi \cdot 1)(\frac{5}{6}) = \frac{1}{6} - \frac{5}{6} = -\frac{2}{3}.$$

   Similarly,

$$E[3^X] = \sum_{x:p(x)>0} g(x)p(x) = 3^0(\frac{1}{6}) + 3^1(\frac{5}{6}) = \frac{1}{6} + \frac{15}{6} = \frac{8}{3}.$$

   and

$$E[\tan^{-1}(X)] = \sum_{x:p(x)>0} g(x)p(x) = \tan^{-1}(0)(\frac{1}{6}) + \tan^{-1}(1)(\frac{5}{6}) = 0 + \frac{\pi}{4}(\frac{5}{6}) = \frac{5\pi}{24}.$$

2. An urn contains 9 balls, 4 of which are red and 5 of which are blue. We draw a ball out of the urn 10 times, taking care to replace the ball and shake up the urn between draws. Let X be the number of times that we draw a red ball.

   (a) What kind of random variable is X and what are the parameters of the random variable?

   Solution. Since we are counting the number of red balls we draw, we are summing up many Bernoullis (each of which is a single draw – and 1,0 is represented by red or not red). Each Bernoulli has parameter $p = \frac{4}{9}$, the probability of getting a red ball. This means X is a **binomial random variable** with parameters $(n, p) = (10, \frac{4}{9})$.

(b) What are $E[X]$ and $Var(X)$?

Solution.

$$E[X] = np = \frac{40}{9}$$
$$Var(X) = np(1-p) = \frac{40}{9} \cdot \frac{5}{9} = 200/81 \sim 2.47$$

(c) What is the probability that $X \leq 3$?

Solution.

$$\begin{aligned}
P(\{X \leq 3\}) &= P(\{X = 0\}) + P(\{X = 1\}) + P(\{X = 2\}) + P(\{X = 3\}) \\
&= \binom{10}{0}\left(\frac{4}{9}\right)^0\left(\frac{5}{9}\right)^{10} + \binom{10}{1}\left(\frac{4}{9}\right)^1\left(\frac{5}{9}\right)^9 \\
&+ \binom{10}{2}\left(\frac{4}{9}\right)^2\left(\frac{5}{9}\right)^8 + \binom{10}{3}\left(\frac{4}{9}\right)^3\left(\frac{5}{9}\right)^7 \\
&\sim .278
\end{aligned}$$

3. Suppose that X is a binomial random variable with parameters n and p. Find $E[X(X-1)(X-2)]$.

Solution. $E[X(X-1)(X-2)] = E[g(X)]$, where $g(X) = X(X-1)(X-2)$. Thus

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x) = \sum_{i=0}^{n} i(i-1)(i-2)p(i),$$

where the sum goes from $i = 0$ to $i = n$ because of the parameter n. The terms inside the sum symbol come from using $g(X)$. Now we note that since this is a binomial random variable with parameter p, we have

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

Therefore,

$$
\begin{aligned}
E[g(X)] &= \sum_{i=0}^{n} i(i-1)(i-2)p(i) \\
&= \sum_{i=0}^{n} i(i-1)(i-2)\binom{n}{i}p^i(1-p)^{n-i} \\
&= \sum_{i=3}^{n} i(i-1)(i-2)\frac{n!}{i!(n-i)!}p^i(1-p)^{n-i} \quad \text{since the first three terms are 0} \\
&= \sum_{i=3}^{n} \frac{n!}{(i-3)!(n-i)!}p^i(1-p)^{n-i} \quad \text{canceling } i(i-1)(i-2) \\
&= n(n-1)(n-2)p^3 \sum_{i=3}^{n} \frac{(n-3)!}{(i-3)!(n-i)!}p^{i-3}(1-p)^{n-i} \\
&= n(n-1)(n-2)p^3 \sum_{j-0}^{n-3} \frac{(n-3)!}{(j)!(n-(j+3))!}p^j(1-p)^{n-(j+3)}, \\
&\quad \text{by setting } j = i-3 \text{, so } i = j+3 \\
&= n(n-1)(n-2)p^3 \sum_{j-0}^{n-3} \binom{n-3}{j}p^j(1-p)^{(n-3)-j} \quad \text{rewriting} \\
&= n(n-1)(n-2)p^3 \cdot (p+(1-p))^{n-3}, \\
&\quad \text{since the expression above is the binomial expansion of } (p+(1-p))^{n-3} \\
&= n(n-1)(n-2)p^3 \cdot 1 = n(n-1)(n-2)p^3
\end{aligned}
$$

4. You have to pay \$100 to play the following game: A fair die is rolled until a 6 appears. If a 6 appears on the nth roll, you win $(\frac{6}{5})^n$ dollars. The game finishes when a 6 appears. Let X be your winnings from the game.

   (a) Prove that $E[X] = \infty$.

   Solution. If X is the winnings, then let Y be the random variable anticipating the turn at which you first get a 6. Then Y is a geometric random variable with parameter $1/6$, and has a probability mass function given by

$$p_Y(i) = \left(\frac{5}{6}\right)^{i-1} \frac{1}{6},$$

and $X = \left(\frac{6}{5}\right)^Y - 100$ The expected value of X is given by thinking of X as a function of Y and using the expected value of Y:

$$E[X] = E[\left(\frac{6}{5}\right)^Y - 100] = \left(\sum_{i\geq 0} \left(\frac{6}{5}\right)^i \left(\frac{5}{6}\right)^{i-1} \frac{1}{6}\right) - 100$$
$$= \sum_{i\geq 0} \frac{1}{5} - 100 = \infty$$

(b) Would you pay a million dollars to play this game?

Solution. If I had lots and lots of money and lots and lots of time, I would play the game because the payoff is really large, on average. However, in practice, I would not, because there is a high chance I would get nothing (and a million dollars is a lot of money to lose). Also, even if I had a lot of money, it may take too long for me to recoup – I would die of old age (perhaps) before being able to make a big win that makes it worth it. Also, it seems to me that in order for this to really work, there is a small pay off of an amount of money that is more than $1,000,000 for each atom in the universe (by a lot). I wouldn't really trust that, small as my chance would be, that if I hit that chance I would win that money!

5. Let X be a Poisson random variable with parameter $\lambda = 3$.

(a) Find $P\{X > 1\}$

Solution.
$P\{X > 1\} = 1 - P\{X \leq 1\} = 1 - (P(\{X = 0\}) + P(\{X = 1\})) = 1 - e^{-3} - 3e^{-3} \sim .8$

(b) Find $E[X(X-1)(X-2)]$.

$$
\begin{aligned}
E[X(X-1)(X-2)] &= \sum_{i=0}^{\infty} i(i-1)(i-2)\frac{e^{-3}3^i}{i!} \\
&= \sum_{i=3}^{\infty} i(i-1)(i-2)\frac{e^{-3}3^i}{i!} \quad \text{since first few terms are 0} \\
&= \sum_{i=3}^{\infty} \frac{e^{-3}3^i}{(i-3!} \text{ by canceling} \\
&= \sum_{j=0}^{\infty} \frac{e^{-3}3^{j+3}}{j!} \text{ by setting } j = i-3 \\
&= 3^3 \sum_{j=0}^{\infty} \frac{e^{-3}3^j}{j!} \\
&= 3^3 \cdot 1 = 27
\end{aligned}
$$

where the last line follows because the sum is just the total probability of a Poisson distribution.

6. Compare the Poisson approximation with the correct binomial probability for the following cases:

(a) $P\{X = 2\}$ when $n = 4, p = \frac{1}{2}$

Solution. Using the pmf for a binomial random variable,

$$p(2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = .375.$$

On the other hand, the Poisson approximation with $\lambda = np = 2$ is

$$p(2) = P(\{X = 2\}) = \frac{2^2e^{-2}}{2!} \sim .2707$$

(b) $P\{X = 2\}$ when $n = 20, p = \frac{1}{10}$.

Solution. Using the pmf for a binomial random variable,

$$p(2) = \binom{20}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} \sim .2851$$

On the other hand, the Poisson approximation with $\lambda = np = 2$ is

$$p(2) = P(\{X = 2\}) = \frac{2^2 e^{-2}}{2!} \sim .2707$$

We notice that the approximation when $n = 20$ and $p = 1/10$ is much better than when $n = 4$ and $p = 1/2$ even though $\lambda = 2$ in both cases.

7. Suppose that a die is rolled until a 6 has appeared five times total (not necessarily in a row). Let X be the number of the roll on which the fifth 6 appears.

   (a) What kind of random variable is X? Make sure to specify any parameters.

   Solution. X is a negative binomial random variable with parameters $r = 5$ and $p = 1/6$.

   (b) What is $E[X]$?

   Solution. The expected value of a negative binomial r.v. is given by $E[X] = r/p$. In this case, $E[X] = 5/(1/6) = 30$.

   (c) What is $Var(X)$?

   The variance of a negative binomial r.v. is given by $Var(X) = \frac{r(1-p)}{p^2}$. In this case, that is $\frac{5*5/6}{1/36} = 150$