MATH 351 Solutions # 4

October 3, 2012

1. THIS QUESTION DID NOT HAVE ENOUGH INFORMATION. I WILL ASSUME THAT, FOR ANY RANDOM LETTER, THERE IS A 50/50 CHANCE IT'S FROM A MALE OR A FEMALE. You get two letters in the mail one day. The first letter you open is from a woman. Find the probability that the other letter is also from a female if

   (a) the mailman does not look at the letters before he gives them to you,
   Solution. We assume that there are infinitely many letters. This means that though you received a letter from a female, there is no new information about whether the next letter is from a male or a female. This implies that there is still a .5 chance that the second letter is from a female.

   (b) the mailman has a ladies first policy and will always hand you a letter from a woman first if there is such a letter.
   Solution. In this case, you do have some new information. In particular, there are 4 possibilities for your two letters: MM, MF, FM and FF. However, since the mailman delivers a female one first no matter what, the only three possibilities are actually MF, FM and FF. Of these, the other letter is from a male in 2 out of the three cases. Thus the answer is 1/3.

2. Each of thirteen people is given 4 cards from a standard deck of 52 cards. What is the probability that each of them has one spade?

   Solution.

   Let E_1 be the event that the ace of spades is in a pile, E_2 is the even that the ace and 2 of spades are in distinct piles, E_3 the probability that the ace of spades, two of spades and 3 of spades are in distinct piles, ... and E_{13} is the event that all spades are in distinct piles.

   Note that $P(E_{13}) = P(E_1E_2 \dots E_{12}E_{13})$. We use the multiplication rule:

   $$P(E_1E_2 \dots E_{12}E_{13}) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \cdots P(E_{13}|E_1E_2 \dots E_{12}).$$

   Now we easily calculate each term: $P(E_1) = 1$ since the ace of spades must be somewhere. $P(E_2|E_1) = 48/51$, since if we know the ace of spades is somewhere, we just have to avoid that pile of 4 cards containing it when we place the two of spades. There are 48 spots not in the pile where the ace of spades was, but 51 spots where the two of spades might be put.

Similarly, for $P(E_3|E_1E_2)$, there are 50 spots to put the 3 of spades since the ace and two are placed somewhere. But if we assume that the ace and two of spades are in distinct piles, and we want to avoid those two piles, there are 44 cards (in the remaining 11 piles) where we could place the 3. Therefore $P(E_3|E_1E_2) = 44/50$.

We continue to find

$$P(E_{13}) = \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \cdot \frac{36}{48} \cdot \frac{32}{47} \cdot \frac{28}{46} \cdot \frac{24}{45} \cdot \frac{20}{44} \cdot \frac{16}{43} \cdot \frac{12}{42} \cdot \frac{8}{41} \cdot \frac{4}{40} = .000106.$$

3. Urn A has 99 red balls and 1 green ball. Urn B has 1 red ball and 99 green balls. An urn is picked at random and a ball is chosen from it. If the ball is red, what is the probability that it came from urn A?

   Solution. Let's let R be the event that we pick a red ball, G be the event we pick a green ball, A be the event that we pick from urn A and B be the event we pick from urn B. Then we are looking for $P(A|R)$.

   We use Bayes' formula to say

$$\begin{aligned} P(A|R) = \frac{P(AR)}{P(R)} &= \frac{P(R|A)P(A)}{P(RA) + P(RB)} \\ &= \frac{P(R|A)P(A)}{P(R|A)P(A) + P(R|B)P(B)} \\ &= \frac{.99 * .5}{.99 * .5 + .01 * .5} = .99 \end{aligned}$$

   Here is some faulty reasoning you might have had: there were 100 red balls, and of these 99 were in urn A, so it seems reasonable that if you have a red ball, there's a 99% chance it is from the red urn. The reason you need to go through Bayes' formula is that you can't be sure *a priori* that there is an equal likelihood of each of these balls. What would have happened if you had a 1/3 chance of picking urn A and a 2/3 chance of picking urn B?

4. A fair die is rolled. If the outcome is i, then a ball is drawn from an urn that has i red balls and 1 blue ball. Suppose that a blue ball was drawn. For each $i = 1, 2, ..., 6$, determine the probability that the outcome of the roll of the die was i.

   Solution. Let E_i be the event that the die rolls an i for $i = 1, 2, \ldots, 6$. Then let B be the even that a blue ball is drawn. We are interested in $P(E_i|B)$.

   We use Bayes' formula:

$$P(E_i|B) = \frac{P(E_iB)}{P(B)} = \frac{P(B|E_i)P(E_i)}{P(BE_1) + P(BE_2) + \cdots P(BE_6)}.$$

Notice that the denominators are equal because $E_1, \dots, E_6$ are mutually exclusive sets. Finally, we use the product rule to rewrite this expression:

$$\frac{P(B|E_i)P(E_i)}{P(BE_1) + P(BE_2) + \cdots P(BE_6)} = \frac{P(B|E_i)P(E_i)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots P(B|E_6)P(E_6)}$$
$$= \frac{\frac{1}{i+1} \cdot \frac{1}{6}}{\frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{7} \cdot \frac{1}{6}}$$

We can cancel the $1/6$, and we obtain the answer $P(E_i|B) = \frac{1}{i+1} \cdot .628$. These values are, in order of increasing i, given by

$$0.313901345, 0.209267564, 0.156950673,$$
$$0.125560538, 0.104633782, 0.089686099.$$

5. Assume $P(E) = 0.5$, $P(F) = 0.3$, and events E and F are independent. Find $P(E \cup F)$.

   Solution. Since the events are independent, $P(EF) = P(E)P(F)$. Therefore, we know all the terms coming from inclusion exclusion:

   $$P(E \cup F) = P(E) + P(F) - P(EF) = .3 + .5 - .3 * .5 = .65$$

6. In a class, there are 5 girls who are at least 20, 7 boys who are under 20 and 9 girls who are under 20. Is it possible for the events at least 20 and girl to be independent if a student is drawn at random from the class?

   Solution. If these two events are independent, then the probability they *both* occur has to be the probability that each one of them occurs. We don't know how many boys at least 20 years old are in the class. Suppose that there are n boys at least 20. Then

   $$P(\text{girl}) = \frac{5+9}{5+9+7+n} = \frac{14}{21+n}.$$

   and

   $$P(\text{at least } 20) = \frac{5+n}{21+n},$$

   while

   $$P(\text{girl at least } 20) = \frac{5}{21+n}.$$

   If they were independent we would need to have

   $$\frac{14}{21+n} \cdot \frac{5+n}{21+n} = \frac{5}{21+n}$$

   Multiply both sides by $(21+n)^2$ and simplify to obtain:

   $$70 + 14n = 105 + 5n \implies 9n = 35,$$

   for which there are no solutions. Therefore, these events could not be independent.