Math 351, Probability
Solutions 3

1. If the letters of the word GOLDIN are arranged randomly in a row, what is the probability that none of the letters in the chosen word is in the same position in which it appears in GOLDIN?

   Solution. Let E_i be the event that the ith letter is in the ith position. We want

   $$P((E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)^c) = 1 - P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6).$$

   We use inclusion-exclusion:

   $$P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6) = \sum_{i=1}^{6} P(E_i) - \sum_{i_1 < i_2} P(E_i E_j) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} E_{i_2} E_{i_3}) - \cdots - P(E_1 \cdots E_6).$$

   We note that $P(E_i) = \frac{5!}{6!}$ for all i, since there are 5! ways to order the other letters in GOLDIN if the ith letter is put in the correct place (and 6! ways to put all the letters). Similarly, $P(E_i E_j) = \frac{4!}{6!}$, $P(E_i E_j E_k) = \frac{3!}{6!}$, $P(E_{i_1} E_{i_2} E_{i_3} E_{i_4}) = \frac{1!}{6!}$, and $P(E_{i_1} E_{i_2} E_{i_3} E_{i_4} E_{i_5}) = P(E_1 \cdots E_6) = \frac{1}{6!}$.Therefore, the answer is

   $$1 - 6 \cdot \frac{5!}{6!} + \binom{6}{2}\frac{4!}{6!} - \binom{6}{3}\frac{3!}{6!} + \binom{6}{4}\frac{2!}{6!} - \binom{6}{5}\frac{1!}{6!} + \frac{1!}{6!}$$
   $$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{53}{144} \approx .368.$$

2. Two cards are chosen at random from a deck of 52 cards. What is the probability that they

   (a) are from different suits?

   (b) are both 4s?

   Solution.

   Part (a). There are $\binom{52}{2} = 1326$ ways to pick a pair of cards from a deck of 52. The ways we can do this from different suits is to subtract off the ways that you can pick two cards with the same suit. Mainly,

   $$\binom{52}{2} - \binom{4}{1}\binom{13}{2} = 1014.$$

Thus the probability is $1014/1326 = .765$

Part (b). There are $\frac{4}{2} = 6$ ways to pick two 4s. Therefore, there is a probability of $6/1326 = .0045$ to pick two 4s.

3. Find a formula for the number of functions from 1,2,...,k **onto** 1,2,...,n.

   Solution.: note that the number of functions *not* hitting i is $(n-1)^k$, since each of k points in the domain can map to each of $n-1$ elements of $\{1, \ldots, n\}$ not equals to i.

   For $i = 1, \ldots, n$, let E_i be the set of functions that do not hit i, and F_i the set of functions that do hit i. Then

   $|F_i| = \text{no. of functions} - |E_i| = n^k - (n-1)^k.$ using inclusion-exclusion.

   Our goal is to find $|F_1 \cap F_2 \cap \cdots \cap F_n|$ as these are the functions that hit each of $1, \ldots, n$, so they are onto. We use inclusion exclusion again:

$$
\begin{aligned}
|F_1 \cap F_2 \cap \cdots \cap F_n| &= n^k - |E_1 \cup E_2 \cup \cdots \cup E_n| \\
&= n^k - n|E_i| - \binom{n}{2}|E_i E_j| + \binom{n}{3}|E_{i_1} E_{i_2} E_{i_3}| \cdots \\
&= n^k - (n(n-1)^k - \binom{n}{2}(n-2)^k + \binom{n}{3}(n-3)^k \\
&- \cdots + (-1)^{n-2}\binom{n}{n-1}(n-(n-1))^k) \\
&= n^k - n(n-1)^k + \binom{n}{2}(n-2)^k - \binom{n}{3}(n-3)^k \\
&- \cdots + (-1)^{n-1}\binom{n}{n-1}(n-(n-1))^k).
\end{aligned}
$$

4. How many people need to be in a room in order that the probability that at least two of them have the same birthday is at least 0.8?

   Solution. We calculate the probability that there are n distinct birthdays among n people, and subtract from 1.

$$
p(n) = \frac{\frac{365!}{(365-n)!}}{365^n},
$$

where the denominator is the number of ways to choose n birthdays (not necessarily distinct) and the numerator is the number of ways to choose n distinct birthdays. This simplifies to

$$\frac{365!}{365^n(365-n)!}.$$

When $n = 35$, this is $\leq .2$, (but when $n = 34$, it is over .02), so at 35 people there's an 80% chance that at least two people share a birthday.

For those who suggested the number was too big, you can do it in Excel by first simplifying $\frac{365!}{(365-n)!}$ for specific choices of n. When $n = 35$, this is PERMUT(365,35), i.e. the product of $365 \cdot 364 \cdots$ for 35 terms.

5. Compute the probability that a bridge hand (13 cards) has at least two cards from every suit.

   Solution. Here we compute the probability of the complement, using inclusion-exclusion, and then subtract from one. Let A_i be the event that a hand has 0 or 1 cards of the ith suit (and not more), for $i = 1, 2, 3, 4$. Then we want to calculate

$$\begin{aligned} P((A_1 \cup A_2 \cup A_3 \cup A_4)^c) &= P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c) \\ &= 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4) \\ &= 1 - \sum_{i=1}^{4} P(A_i) + \sum_{i<j} P(A_iA_j) - \sum_{i<j<k} P(A_iA_jA_k) + P(A_1A_2A_3A_4) \end{aligned}$$

   We find that $P(A_i) = [\binom{39}{13} + \binom{13}{1}\binom{39}{12}]/\binom{52}{13}$ for all i.

   And, $P(A_iA_j) = [\binom{26}{13} + \binom{26}{11}\binom{13}{1}\binom{13}{1} + \binom{26}{12}\binom{13}{1} \cdot 2]/\binom{52}{13}$ for all $i \neq j$.

   For the three-way intersection, $P(A_iA_jA_k) = [\binom{13}{13} + \binom{13}{10}\binom{13}{1}^3 + \binom{13}{11}\binom{13}{1}^2\binom{3}{2} + \binom{13}{12}\binom{13}{1}\binom{3}{1}]/\binom{52}{13}$.

   Finally, $P(A_1A_2A_3A_4) = 0$ since you can't have 0 or 1 cards in all four suits.

   We stick this into the formula above to get the answer (which I didn't have time to compute).

6. A coin is flipped until heads has appeared four times. What is the probability that the fourth head appears on the tenth flip?

Solution. This is the same as the probability of getting *exactly* 3 heads in the first 9 flips, and then getting H on the 10th flip.

$$P(3 \text{ H in 9 flips}) = \binom{9}{3}(.5)^9,$$

which means the answer $\binom{9}{3}(.5)^{10} \approx .08$.

7. A blue die and a red die are rolled. What is the probability that

   (a) their sum is at least 6, given that the red die is even?

   (b) the red die is 3, given that their sum is at least 6?

   Solution.

   Part (a). Let E be the event the red die is even, and F the event that the sum is ≥ 6. Then

$$P(F) = 1 - P(\text{sum is } < 6) = 1 - (\frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36}) = \frac{26}{36}.$$

   Furthermore, $P(E) = \frac{1}{2}$, and finally $P(EF) = \frac{3}{36} + \frac{5}{36} + \frac{6}{36} = \frac{14}{36}$. The number $\frac{3}{36}$ is the probability the red die is 2 (and F), $\frac{5}{36}$ is the probability the red die is 4 (and F), and $\frac{6}{36}$ is the probability it is 6 (and F). Therefore,

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{14/36}{1/2} = \frac{14}{18} = \frac{7}{9}.$$

   Part(b). Let G be the event the red die is 3. Then $P(G) = 1/6$. $P(FG) = \frac{4}{36} = \frac{1}{9}$ as the other die must be $3, 4, 5, 6$. Finally

$$P(G|F) = \frac{P(GF)}{P(F)} = \frac{1/9}{26/36} = \frac{2}{13}.$$

8. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are hearts. Let A be the event that the ace of hearts is chosen, and H the event that at least one heart is chosen.

   (a) Find $P(B)$.

   (b) Find $P(B|A)$.

   (c) Find $P(B|H)$.

   Solution. We essentially did this in class.