MATH 351
Solutions #2

1. Two dice are thrown. Let E be the event that the sum of the dice is even, let F be the event that at least one of the dice lands on 6 and let G be the event that the numbers on the two dice are equal. Find $P(E), P(F), P(G), P(EF), P(FG), P(EG)$.

   Solution. Out of 36 possible rolls, the ones that the sum of the dice is even are those in which both dice are either even or odd. There are 3 odd numbers on the first die, and for each of these, there are 3 odds on the second – resulting in 9 possible odd/odd combinations. Similarly, there are 9 ways to get both even numbers. This results in $P(E) = 18/36 = 1/2$. For $P(F)$, note that if the first die is a 6, the second could be anything (6 possibilities) and if the second is a 6, the first could be anything (another 6 possibilities), but the pair $(6, 6)$ will be listed twice – so there are only 11 ways that this could happen. $P(F) = 11/36$. There are 6 ways to get doubles, so $P(G) = 6/36 = 1/6$. The set $EF := E \cap F = \{(2, 6), (4, 6), (6, 6), (6, 4), (6, 2)\}$, a set of size 5, so $P(EF) = 5/36$. $E \cup F$ is the case that either the dice are the same parity (18 ways), or at least one lands on a 6 (11 ways) – so we can add up 18+11 and subtract off the common elements of this set (which we will have overcounted this way).

   $$P(E\cup F) = P(E)+P(F)-P(EF) = (18+11-5)/36 == 24/36 = 1/3.$$

   Similarly, $P(FG) = 1/36$ (doubles and at least one six implies both are 6), so

   $$P(F \cup G) = P(F) + P(G) - P(FG) = (11 + 6 - 1)/36 = 16/36 = 4/9.$$

2. A card game requires each person to receive 4 cards from a standard deck of 52.

   (a) How many hands are possible?
   Ans. $\binom{52}{4} = 270725$

   (b) What is the probability that a hand will contain 4 of a kind?
   Ans. There are 13 possible hands that contain 4 of a kind (you can think of this as $\binom{13}{1}\binom{4}{4}$, since you first choose a face value, then

you choose 4 of the 4 cards of that face value). The probability is $13/270725 = 0.0004802$.

(c) What is the probability that a hand will contain 3 of a kind?

Ans. Again you choose one face value (13 ways), then choose 3 of the 4 cards of that face value. Finally, there are 48 cards remaining that are not the face value you have chosen. That will be your 4th card.

$$\frac{13 * \binom{4}{3} * 48}{\binom{52}{4}} = .0092.$$

3. Suppose we roll 4 6-sided dice simultaneously. Show that

   (a) $P(\text{no two alike}) = \frac{5}{18}$

   (b) $P(\text{two pair}) = \frac{5}{72}$

   (c) $P(\text{3 alike}) = \frac{5}{54}$.

4. An urn contains 3 red, 5 blue and 7 green balls. A set of 3 of the balls is randomly selected.

   (a) What is the probability that all 3 of the selected balls are red?

   Ans. There are 15 balls total, and $\binom{15}{3}$ ways to pick 3 of them. Only one of these is a choice of 3 red balls (or $\binom{3}{3}$). this makes for a total of $1/\binom{15}{3} = 1/455 = .0022$

   (b) What is the probability that all 3 have different colors?

   Ans. The number of ways to pick 3 distinct balls is $\binom{3}{1}\binom{5}{1}\binom{7}{1} = 105$. Thus the probability is

$$\binom{3}{1}\binom{5}{1}\binom{7}{1}/\binom{15}{3} = 105/455 = .231.$$

5. A school offers Spanish, French and German language classes. 100 students take at least one of the three languages. 39 take Spanish, 38 take French, and 37 take German. 72 take Spanish or French, 2 take French and German, 4 take Spanish and French but not German.

   (a) How many students take all three languages?

Ans. We are looking for $|SFG| = |S \cap F \cap G|$. We use inclusion exclusion, noting that SFG and SFG^c are disjoint sets.

$$\begin{aligned} |S \cup F| &= |S| + |F| - |SF| \\ 72 &= 39 + 38 - |SF| \implies |SF| = 5. \\ |SF| &= |SFG \cup SFG^c| = |SFG| + |SFG^c|, \\ 5 &= |SFG| + 4 \implies |SFG| = 1. \end{aligned}$$

(b) How many take only Spanish?

Ans. We are looking for $|S \cap F^c \cap G^c| = |F^c \cap G^d|$ (the reason these are the same is that our pool of students includes anyone taking a language – so if you're not taking French or German, you *are* taking Spanish.). Note that $F^c \cap G^c = (F \cup G)^c$. Now, $|F \cup G| = |F| + |G| - |FG| = 38 + 37 - 2 = 73$. Therefore, $|F \cup G|^c = 100 - 73 = 27$.

6. A pair of dice is rolled until either **the two numbers on the dice agree** or **the difference of the two numbers on the dice is 1** (such as a 4 and a 5, or a 2 and a 1). Find the probability that you roll two dice whose numbers agree before you roll two dice whose numbers differ by 1.

   Hint: Let E_n denote the event that the numbers agree on the nth roll and that on the first $n-1$ rolls the dice neither agree nor differ by one. Compute $P(E_n)$ and argue that $\sum_{n=1}^{\infty} P(E_n)$ is the desired probability. Then calculate this sum.

   Solution. We follow the suggestion for E_n. Then $P(E_n)$ is the probability that we do *not* get doubles or numbers differing by one in the first $n-1$ rolls, but we *do* get doubles or numbers differing by one on the nth roll.

   There are 6 doubles $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$ and 10 pairs with differences of 1 $\{(1,2),(2,3),(3,4),(4,5),(5,6),(6,5),(5,4),(4,3),(3,2),(2,1)\}$. The consists of 16 of the 36 possible outcomes $(\frac{4}{9})$ when we roll two dice. Technically, the sample space $\mathcal{S}_n$ is the set of all pairs of n rolls of the two dice (note that $\mathcal{S}_n$ depends on n). This means that, for each roll, there are 20 outcomes in which neither a doubles nor pairs where the dice different by one occur. This occurs on any specific roll with probability $20/36 = 5/9$. On the other hand, there are 6 outcomes in which the pairs happen ($\frac{1}{6}$ probability)Therefore, the probability we get neither ending roll on the first $n-1$ rolls, and a double on the nth roll iw

$$P(E_n) = \left(\frac{5}{9}\right)^{n-1} \frac{1}{6}.$$

Now to find the probability of this ending the game with doubles, we simply add these probabilities up for all n. Let E be the event that we end the game on doubles. Then

$$\begin{aligned} P(E) &= \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^{n-1} \frac{1}{6} \\ &= \frac{1}{6}\left(\sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^{n-1}\right) \\ &= \frac{1}{6}\left(1 + \sum_{n=1}^{\infty} \left(\frac{5}{9}\right)^{n}\right) \qquad \text{Notice reindexing} \\ &= \frac{1}{6} \times \frac{1}{1-(5/9)} = 9/24 = \frac{3}{8}. \end{aligned}$$