MATH 351
Solutions #1

1. How many different 8-digit reservation codes by an airline are possible if the first two places are occupied by letters, the next three places are occupied by numbers, the sixth place may be a letter or a number, and the last two digits must be letters?

   Solution. There are 26 letters possible for each letter "spot" and 10 numbers possible for each number "spot." One spot has 36 possibilities since it can be a number of a letter. By the generalized basic principle of counting, we simple take the product of the number of possibilities of each entry:

   $$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 36 \cdot 26 \cdot 26 = 26^4 \cdot 10^3 \cdot 36 = 16{,}451{,}136{,}000$$

2. How many functions are there from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$? How many of these are one-to-one?

   Solution. A function is an assignment of an integer $1, 2, \ldots, n$ to each of the numbers $1, \cdots, k$ in the domain. In other words, a function f is specified by the sequence $f(1), \cdots, f(k)$. Since each $f(i)$ could be one of n numbers for each of $i = 1, \cdots, k$. By the Basic Principle, there are $n \cdot n \cdots n$ (k times) $= n^k$ possible different assignments total.

   A one-to-one map is INJECTIVE, which immediately implies that $k \leq n$ (else there are no one-to-one functions). In this case, there are n possibilities for the value of $f(1)$, $n - 1$ possibilities for $f(2)$ since the first number cannot be reused, $n-2$ choices for $f(3)$ since the first two numbers cannot be repeated, etc. In total (using the Basic Principle of Counting), there are $n \cdot n - 1 \cdot n - 2 \cdots n - k + 1 = \frac{n!}{(n-k)!}$ possibilities. Notice that there are k terms in the product.

3. How many different letter arrangements can be formed from the letters of the word MATHEMATICSISCOOL? How many of these do not have two consecutive A's?

   Solution. Similar to a problem in class, let x be the number of ways to put these letters in some arrangement. There are 17 letters in all, but there are repeats. The letters M, A, T, I, S, C and O each repeat exactly twice. If we could distinguish among the different orders to put each of the pairs of letters, there would be 17! ways to write the letters. When we can't distinguish among the letters in each pair, there are two ways we could have written the sequence if we could have distinguished. Thus we have the relationship

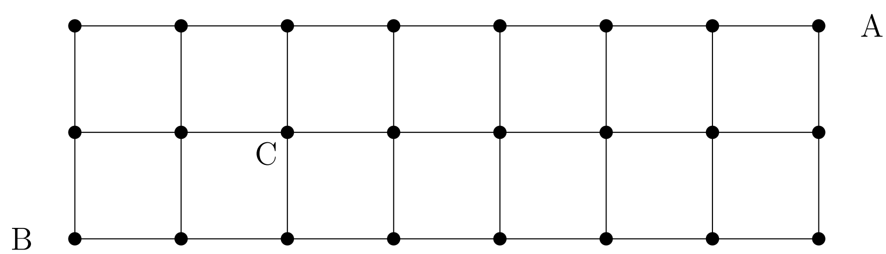
$$17! = x \cdot (2!)^7, \text{ resulting in } \quad x = \frac{17!}{2^7}$$

possibilities.

To get those in which we do not allow two consecutive As, we can count the number of times the two As appear together. Consider the pair AA to be one letter. In this case, there are only 16 letters to play with, and there are $\frac{16!}{2^6}$ different words made out of the letters in MTHEMTICSISCOOLAA, where AA is one letter. Thus the possible arrangements in which the As are not together are:

$$\frac{17!}{2^7} - \frac{16!}{2^6}.$$

4. Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step down, or one step to the left at each move. This procedure is continued until the point labelled B is reached. How many different paths from A to B are possible? How many of these paths do *not* go through point C?



Solution. To get from A to B, you have to go down twice and left 7 times. Every path consists of 9 moves, of which two are downward moves and 7 are left moves. You can move down at any time if you haven't already gone two moves. Therefore, there are $\binom{9}{2} = 36$ paths total. The two that you are choosing are the moves in which you go down (for the other moves, you go across).

Each path through C is a path from A to C and a path from C to B. To get from C to B you have to go one move down and 2 to the left (3 moves total), and there are $\binom{3}{1}$ ways to do this (you pick 1 of 3 moves to be the one in which you'll move down). To get from A to C, you have to go down 1 and left 5 moves, for a total of $\binom{6}{1}$ paths (you choose one in 6 moves in which you'll move down). Thus there are $3 \cdot 6 = 18$ paths through C.

Therefore, the number of paths that avoid C are $36 - 18 = 18$.

5. If 6 Geometry books, 4 Probability books and 2 Algebra books are to be arranged on a shelf, how many arrangements are possible if books of the same subject must be next to each other?

   Solution. In this case, the individual books can be distinguished, but they have to be put together. First, we consider each subject within mathematics as a group, and order the groups. There are 3 groups (geometry, probability and algebra), so there are $3! = 6$ ways to order these groups. Then for each way of ordering the three groups, there are $6!$ ways to order the geometry books, $4!$ ways to order the probability books, and 2 ways to order the algebra books, resulting in

$$3! \cdot 6! \cdot 4! \cdot 2 = 207,360$$

   ways of ordering all of them.

6. A certain club contains 10 men and 10 women. A six member committee consisting of 2 men and 4 women must be chosen from the club. How many committees are possible? How many committees are possible if Mr. and Mrs. Smith refuse to serve together on the committee?

   Solution. We have to choose 2 among the 10 men, and 4 among the 10 women, so there are

$$\binom{10}{2} \cdot \binom{10}{4} = 45 \cdot 210 = 9,450.$$

   ways to make a group. If Mr. and Mrs. Smith refuse to work together, then we count the ways to include both of them, and exclude them from the total number. There are $\binom{9}{1}$ to choose 1 man in addition to

Mr. Smith, and $\binom{9}{3}$ ways to choose 3 women in addition to Mrs. Smith, resulting in $9 \cdot 84 = 756$ ways to pick Mr. and Mrs. Smith among all the chosen people, and inlcude them both. Therefore, the number of committees containing at most one of the two is

$$9,450 - 756 = 8,694.$$