SOLUTIONS

Math 351, Probability
Exam 1
October 4, 2012

Directions. Please write your name at the top of the page. Your answers should be in simplest form possible without a calculator. You may leave answers as fractions, differences of fractions, factorials, choose, etc. You are encouraged only to simplify when obvious and necessary. Please BOX YOUR ANSWERS. You must show work for credit. There are 100 points on the exam, so pace yourself accordingly.

1. *(14 points)*

   (a) (8 points) State the binomial theorem.

For $n \geq 0$ integers $\quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

   (b) (6 points) Use the binomial theorem to prove that

$$\sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k} = 1.$$

Let $x = .5 = y$.

Then $(x+y)^n = 1 \quad \forall\, n \geq 0$

$$\Rightarrow \quad 1 = \sum_{k=0}^{n} \binom{n}{k} .5^k .5^{n-k} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n \binom{n}{k}.$$

2. *(10 points)* Three 6-sided dice are rolled until either a triple occurs (3 of the same value), or the numbers 1,2,3 occur at the same time. What is the probability that the game finishes with a triple occurring? Note that $6 \cdot 6 \cdot 6 = 216$.

There ~~are~~ is $\frac{6}{216} = \frac{1}{36}$ probability ~~chance~~ to get a triple, and $\frac{6}{216} = \frac{1}{36}$ probability ~~chance~~ to get $\{1,2,3\}$ showing ($= 3!/216$).

Thus there is a $\frac{34}{36} = \frac{17}{18}$ chance of neither.

The chance the game finishes on the n^{th} roll with a triple is given by $\underbrace{\left(\frac{17}{18}\right)^{n-1}}_{\text{chance of neither}} \underbrace{\left(\frac{1}{36}\right)}_{\text{chance of finishing with triple}}$.

Thus overall, the probability of finishing with a triple is

$$\sum_{n=0}^{\infty} \left(\frac{17}{18}\right)^n \left(\frac{1}{36}\right) = \frac{1}{36} \sum_{n=0}^{\infty} \left(\frac{17}{18}\right)^n = \frac{1}{36}\left(\frac{1}{1-\left(\frac{17}{18}\right)}\right) = \frac{1}{36} \frac{1}{1/18} = \boxed{\frac{1}{2}}.$$

[Alternatively, you could note that, at each stage there is 1/36 chance of finishing with triples, and 1/36 chance of finishing with $\{1,2,3\}$, or the game restarts, and is independent of previous game. ~~Thus~~ You can think of 12 ways to finish the game, of which 6 are triples. These are equally likely, and since the next "round" is independent of the first, the chance is $\boxed{\frac{1}{2}} = \frac{6}{12} = \frac{1/36}{2/36}$

3. *(10 points)* Ten people from ten different countries are seated around a round a table at the George Mason Southside Dining Hall. Two of the countries represented are Spain and Italy. What is the probability that the students from these countries are seated together?

Since people sit around the table, there are 9! ways to place people relative to each other (you can think of a rotation of the entire table as irrelevant). You can seat them together by considering them as one "unit". As a unit of 2, there are 8! ways to seat them, and 2 ways to order Spain & Italy within the pair. This means the probability is $\frac{2 \cdot 8!}{9!} = \boxed{\frac{2}{9}}$.

Alternatively, you could seat Italy first, and then note that there are 9 remaining spots for Spain, of which 2 are next to Italy. Thus the prob. is $\boxed{\frac{2}{9}}$.

Alternatively, you could imagine 10! ways to order people, as if you "cut" the table at some point & labeled the seats 1, 2, ..., 10. From this point of view, the ways to put Italy and Spain together are given by 9 ways to choose 2 spots 12, 23, 34, ..., 910 and then 2 ways to order them, ~~and~~ with 8! ways to order the others, for a total of $2 \cdot 9 \cdot 8!$ and finally for the (10,1) positions, there are an additional $2 \cdot 8!$ ways to put Spain & Italy in these 2 positions. This totals $2 \cdot 9 \cdot 8! + 2 \cdot 8!$ ways, for a probability of

$$\frac{2 \cdot 9 \cdot 8! + 2 \cdot 8!}{10!} = \frac{2 \cdot 10 \cdot 8!}{10!} = \boxed{\frac{2}{9}}$$

4. *(10 points)* Three students meet up at Jane's Juicy Juice. Without paying attention to each other, they each randomly order one of 20 variations of juice. What is the probability that two or more of them ordered the same drink?

Let E be the event that two or more order the same drink.
Then E^c is the event that none have ordered the same drink.

Then $P(E) = 1 - P(E^c)$.

We calculate $P(E^c) = \dfrac{20 \cdot 19 \cdot 18}{20^3}$ ← number of ordered triples of distinct drinks / ← number of ordered triples of drinks

Thus $P(E) = 1 - \dfrac{19 \cdot 18}{20^2}$.

5. *(10 points)* Urn A contains 198 red balls and 2 white balls. Urn B contains 1 red ball, and 99 white balls. A computer is programmed to pick Urn A with probability 1/3 and Urn B with probability 2/3. The computer selects an urn, and then a ball is drawn. If the ball drawn is white, what is the probability that it came from Urn B?

Let A = event urn A is chosen
B = " B . . .
R = event red ball is chosen
W = event white ball is chosen

We want $P(B|W)$.

$$P(B|W) = \frac{P(WB)}{P(W)}$$

$$= \frac{P(W|B)\,P(B)}{P(WB) + P(WA)}$$

$$= \frac{P(W|B)\,P(B)}{P(W|B)P(B) + P(W|A)P(A)}$$

$$= \frac{.99 \cdot 2/3}{.99 \cdot 2/3 + .01 \cdot 1/3} = \frac{.66}{.66 + .0033\ldots} \approx .995$$

6. *(16 points)* Four professors, one of whom is Dr. Goldin, are playing poker. They are each dealt 5 cards from one normal 52-card deck of cards.

   (a) What is the probability that Dr. Goldin has a full-house? A full-house is 3 of a kind, and 2 of another kind.

There are $\binom{52}{5}$ hands that Dr. Goldin could have. We have to count how many of these hands ~~have~~ are a full house.

There are 13 ways to ~~a~~ choose a face value, then $\binom{4}{3}$ ways to choose 3 of the 4 cards of this face value. Then 12 ways to pick a second ~~face~~ value, and $\binom{4}{2}$ ways to pick 2 of this 2^{nd} face value. In total, there are $13\cdot\binom{4}{3}\cdot 12\cdot\binom{4}{2}$ hands.

Alternatively, there are $\binom{13}{2}$ ways to pick 2 face values, and then $\binom{4}{3}\cdot\binom{4}{2}$ ways to pick the triple & pair, but then 2 ways to choose which is the triple & which is the pair, for a total of $\binom{13}{2}\binom{4}{3}\binom{4}{2}\cdot 2$ ways.

The probability is therefore $\dfrac{13\cdot 12\cdot 4\cdot 6}{\binom{52}{5}}$ of getting a full house.

   (b) Dr. Goldin gets 3 of a kind, but her other two cards do not match the other cards nor each other. She discards two cards to try to get a full house or four of a kind, and receives 2 new cards from the pile. (The pile does not contain any discarded cards in it). What is the probability that she succeeds in getting a full house or four of a kind?

Here the key thing is that in terms of which cards Dr. Goldin might receive, *it doesn't matter that some cards have been distributed to others *, provided we don't know what others have in their hands. Thus we are only concerned about the 47 cards that Dr. Goldin has not seen. We know of these 47 cards, that one card matches the 3 in her hand. Of the other 46 cards there are 6 cards (3 of each face value) matching the 2 cards she discards.

Therefore, the probability of picking 2 cards and getting 4 of a kind is $\dfrac{1\cdot 46}{\binom{47}{2}}$. the other card

The probability of picking 2 cards and getting a pair is $\dfrac{\binom{3}{2}\cdot 2+\binom{4}{2}\cdot 10}{\binom{47}{2}}$

(where $\binom{3}{2}\cdot 2$ is the ways of getting a ~~pr~~ pair with face value of the face values she threw away, and $\binom{4}{2}(10)$ is the ways of getting a pair with a face value she has not yet seen). Thus the ~~~~ probability of getting a 4 of a kind or a full house is $\dfrac{46+\binom{3}{2}\cdot 2+\binom{4}{2}\cdot 10}{\binom{47}{2}}$.

7. *(20 points)* Ten people are dealt 5 cards each from a standard deck of 52, with the remaining cards set aside.

   (a) What is the probability that the first person has exactly 1 heart? (Remember a 'hand' is the set of card a person holds).

   $$\frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}}$$

   ← 13 hearts, of which 1st person holds 1. The 4 other cards are chosen among the 39 non-heart, to have exactly one heart.

   ← # of hands 1st person could have

   (b) What is the probability that the second hand has exactly 1 heart, given that the first person has exactly one heart?

   The first person holds 1 Heart & 4 non-hearts. There are 12 remaining hearts & 35 remaining non-hearts.
   The possible hands are $\binom{47}{5}$ as 5 have been distributed to person 1.
   There ~~are~~ is $\frac{\binom{12}{1}\binom{35}{4}}{\binom{47}{5}}$ probability.

   (c) What is the probability that the third hand has exactly 1 heart, given that the first two people have exactly one heart each?

   $$\frac{\binom{11}{1}\binom{31}{4}}{\binom{42}{5}}$$

   (d) What is probability that the last person has two hearts, given the first 9 people have exactly one heart each?

   $$\frac{\binom{4}{2}\binom{3}{3}}{\binom{7}{5}}$$

   ← 4 hearts left, 3 non hearts left, after the 1st 9 people have received their hands.

   ← 7 cards remaining after $9 \times 5 = 45$ cards have been distributed

   (e) What is the probability that the first nine people have one heart each, but the last person has exactly two hearts? Justify your answer. You are encouraged to use your answers to the previous parts.

   Let H_i = prob. i^{th} person has exactly 1 heart, $i = 1, 2 \ldots, 9$
   and H_{10} = prob. 10^{th} person has 2 hearts.
   We want $P(H_1 \cdots H_{10})$.

   $$P(H_1 \cdots H_{10}) = P(H_1)\,P(H_2 \mid H_1)\,P(H_3 \mid H_1 H_2) \cdots P(H_{10} \mid H_1 \cdots H_9)$$

   $$= \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} \frac{\binom{12}{1}\binom{35}{4}}{\binom{47}{5}} \frac{\binom{11}{1}\binom{31}{4}}{\binom{42}{5}} \cdots \frac{\binom{4}{2}\binom{3}{3}}{\binom{7}{5}}.$$

8. *(10 points)* Consider an urn containing four balls, numbered 110, 101, 011 and 000, from which one ball is drawn at random. For k = 1, 2, 3. Let A_k be the event of drawing a ball with 0 in position k. Show the events are pairwise independent, but not mutually independent.

$\cancel{P(A_k) =}$

$A_1 = \{011, 000\}$

$A_2 = \{101, 000\}$ $\qquad P(A_i) = \frac{1}{2} \quad i = 1, 2, 3$

$A_3 = \{110, 000\}$ $\qquad (= \frac{2}{4})$

$S = \{011, 101, 110, 000\}$

$A_1 A_2 = A_2 A_3 = A_1 A_3 = \{0,0,0\}$

$P(A_i A_j) = \frac{1}{4}$ with $i \neq j$.

Then $P(A_i)P(A_j) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A_i A_j)$ for $i \neq j$ implies that they are pairwise independent.

However,

$A_1 A_2 A_3 = \{0,0,0\} \Rightarrow P(A_1 A_2 A_3) = \frac{1}{4}$.

However

$P(A_1) P(A_2) P(A_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq 1/4 = P(A_1 A_2 A_3)$

So A_1, A_2, A_3 are not mutually independent.