1. Section 2.4 # 6 (a), (g)

(a) Let $a_n = \left\{ \frac{n+1}{2n+1} \right\}$. The sequence $a_n$ is clearly positive for all $n \in \mathbb{N}$. Thus, $a_n$ is bounded below by 0.

We show that $a_n$ is also decreasing. Consider the ratio $\frac{a_{n+1}}{a_n}$. If we show it is less than 1, then the sequences is strictly decreasing.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{2n+3}}{\frac{n+1}{2n+1}} = \frac{(n+2)(2n+1)}{(2n+3)(n+1)} = \frac{2n^2 + 5n + 2}{2n^2 + 5n + 3}$$

Since $2n^2 + 5n + 3 > 2n^2 + 5n + 2$ for all $n \in \mathbb{N}$, it follows that the numerator is smaller than the denominator, or $a_{n+1} < a_n$. Thus $\{a_n\}$ is decreasing and bounded below, which implies it converges.

(g) Let $a_n = \left\{ \frac{(-1)^n+1}{n} \right\}$. Clearly $|a_n| = \left| \frac{1}{n} \right| \leq 1$ for all $n \in \mathbb{N}$. Therefore $\{a_n\}$ is bounded by 1.

However $\{a_n\}$ is not monotone nor eventually monotone, since for every $n^*$ there exists $n_1 > n_2 \geq n^*$ with $n_1$ odd and $n_2$ even, so that $a_{n_1} > a_{n_2}$, and on the other hand, there exists $n_1 > n_2 \geq n^*$ with $n_1$ even and $n_2$ odd, so that $a_{n_1} < a_{n_2}$. (Therefore the sequence is neither increasing nor decreasing.)

Here is another proof that it’s not monotone due to Robert Allen:

$\forall n_1, n_2 \in \mathbb{N}, n_1 \leq n_2 \not\Rightarrow a_{n_1} \leq a_{n_2}$ since $a_1 = 1$ and $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$.

Thus, $\{a_n\}$ is not monotonic.

Since $\{a_n\}$ is not monotonic, we can not apply any Theorems on bounded monotonic sequences. However, we know that $\{a_n\}$ converges to 0.
2. Section 2.4 #11 (c)
(proof due to R. Allen) First, we will show inductively that \( \{a_n\} \) is strictly increasing. Notice that 
\[
a_2 = \sqrt{\sqrt{2}} = \sqrt{2} > \sqrt{2} > 1.
\]
Assume 
\[
a_{k+1} > a_k \text{ for some } k \in \mathbb{N}.
\]
\[
a_{k+2} = \sqrt{2a_{k+1}} > \sqrt{2 \sqrt{a_k}} = a_{k+1}.
\]
Thus, \( \{a_n\} \) is strictly increasing.

Next, we will show inductively that \( \{a_n\} \) is bounded above by 3. \( a_1 = \sqrt{2} < 3 \).
Assume 
\[
a_k < 3 \text{ for some } k \in \mathbb{N}.
\]
\[
a_{k+1} = \sqrt{2a_k} < \sqrt{2 \cdot 3} < \sqrt{6} < 3.
\]
So, \( \{a_n\} \) is bounded above by 3.

Since \( \{a_n\} \) is strictly increasing and bounded above, \( \{a_n\} \) converges.

Now we will prove that \( \{a_n\} \to 2 \). Assume \( \lim_{n \to \infty} a_n = A \).

We know that
\[
\lim_{n \to \infty} a_{n+1} = A \text{ as well.} \lim_{n \to \infty} 2a_n = \lim_{n \to \infty} \sqrt{2a_n} = \\
\sqrt{2} \lim_{n \to \infty} \sqrt{a_n} = \sqrt{2} \sqrt{\lim_{n \to \infty} a_n} = \sqrt{2} \sqrt{A} = \sqrt{2A}.
\]
Thus, 
\[
A^2 = 2A \iff A^2 - 2A = 0 \iff A(A - 2) = 0 \iff A = 0 \text{ or } A = 2.
\]
Since \( a_1 > 0 \) and \( \{a_n\} \) is strictly increasing, it can not converge to 0.

\[
\therefore \lim_{n \to \infty} a_n = 2.
\]

3. Section 2.5 #2 (a), (c)
(a) We note that \( 0 < |x - 1| < 3 \) implies \( 0 < x - 1 < 3 \) or \( -3 < x - 1 < 0 \),
which implies \( 1 < x < 4 \) or \( -2 < x < 1 \). Any \( s \in (1, 4) \) is an accumulation point: let \( \epsilon > 0 \).
Let \( \epsilon_1 = \min\{\epsilon, s - 1, 4 - s\} \).
Note \( \epsilon_1 > 0 \). Then \( s - \frac{\epsilon_1}{2} \in (1, 4) \), and \( s - \frac{\epsilon_1}{2} \in N^s(\epsilon) \) (the deleted neighborhood of \( s \)),
as \( 0 < \frac{\epsilon_1}{2} < \epsilon \).
Similarly, any \( s \in (-2, 1) \) is an accumulation point.

We note that the points \(-2, 1, 4\) are also accumulation points. Let \( s \) be one of these points. Then for any \( \epsilon > 0 \), we claim there are points in the deleted \( \epsilon \)-neighborhood of \( s \).
Consider \( s = -2 \). If \( \epsilon < 3 \), then certainly \( -2 + \frac{\epsilon}{2} \in N^s(\epsilon) \).
If \( \epsilon \geq 3 \), then \( 2.1 \in N^s(\epsilon) \). Thus \(-2\) is an accumulation point.
Similarly, \( s = 1 \) and \( 4 \) are accumulation points by carrying out the same argument.

Thus, the set of accumulation points of \( S \) is \([-2, 4] \).
Section 2.5 #2 (c)
Let \( \{a_n\} = \begin{cases} 0, & \text{if } n \text{ is odd.} \\ \frac{n}{n+1}, & \text{if } n \text{ is even} \end{cases} \)

First we show that \( \frac{n}{n+1} \) is strictly increasing and \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \). (Proof due to R. Allen) It is clear that \( \{a_n\} \) is a subsequence of \( \{a_n\} \). Choose \( \epsilon > 0 \).

Let \( n^* \in \mathbb{N} \) st. \( n^* \geq 1 \). Thus \( \forall n > n^*, n > \frac{1}{\epsilon} - 1 \iff n + 1 > \frac{1}{\epsilon} \iff \frac{1}{n+1} < \epsilon \iff \frac{1}{n+1} < \epsilon < \frac{1}{n} < \epsilon < \frac{1}{n-1} < \epsilon \).

We show by contradiction that there are no accumulation points in the interval \((0, 1)\). Suppose \( s \in (0, 1) \) were an accumulation point. Suppose \( s < 2/3 \). Then let \( \epsilon = \min\{s, 2/3 - s\} \). If \( s = 2/3 \), choose \( \epsilon = 4/5 - 2/3 \).

Now we will show that \( 1 \) is an accumulation point of \( S \). Since \( \{a_{2n}\} \to 1 \), \( \forall \epsilon > 0, \exists n^* \in \mathbb{N} \) s.t. \( \forall n \geq n^* \), \( |a_{2n} - 1| < \epsilon \). Since \( a_{2n} \neq 1 \) for any \( n \), we have \( a_{2n} \in N^*_\epsilon(1) \) for any \( n \geq n^* \). Thus \( 1 \) is an accumulation point of \( S \).

\( \therefore \{1\} \) is the set of accumulation points of \( S \).

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4. Section 2.5 #7 (a), (c)

(a)
Let \( \{a_n\} = \frac{n+1}{m} \).

\[ |a_m - a_n| = \left| \frac{m+1}{m} - \frac{n+1}{n} \right| = \left| \frac{mn+n-mn-m}{mn} \right| = \left| \frac{n-m}{mn} \right| = \frac{1}{m} - \frac{1}{n} \leq \frac{1}{m} + \frac{1}{n} \]

Choose \( \epsilon > 0 \). Then since \( \left\{ \frac{1}{n} \right\} \) converges to \( 0 \), \( \exists n^*_1 \in \mathbb{N} \) s.t. \( \forall n \geq n^*_1, \ |\frac{1}{n}| \leq \frac{\epsilon}{2} \).

Similarly, since \( \left\{ \frac{1}{n} \right\} \to 0 \), \( \exists n^*_2 \in \mathbb{N} \) s.t. \( \forall n \geq n^*_2, \ |\frac{1}{n}| \leq \frac{\epsilon}{2} \).

Thus, \( \forall m, n > \max\{n^*_1, n^*_2\}, \ |a_m - a_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \).

\( \therefore \{a_n\} \) is Cauchy.
Section 2.5 #7 (c)
Choose \( \epsilon > 0 \). We want to show there exists \( n^* \) such that \( n, m \geq n^* \) implies
\[
\left| \sum_{i=1}^{m} \frac{1}{i^2} - \sum_{j=1}^{n} \frac{1}{j^2} \right| < \epsilon \Rightarrow |a_m - a_n| < \epsilon.
\]
Without loss of generality, assume that \( m > n \). We note that
\[
\sum_{i=1}^{m} \frac{1}{i^2} - \sum_{j=1}^{n} \frac{1}{j^2} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2}.
\]
We use the hint suggested in the back of the book, that, for \( n > 1 \),
\[
\frac{1}{n} - \frac{1}{n-1} < \frac{1}{n-1} - \frac{1}{n}.
\]
This follows by simplifying the right hand side of this inequality and inverting it. Then
\[
\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} < \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{1}{m-1} - \frac{1}{m} = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.
\]
Thus let \( n^* > \frac{1}{\epsilon} \) and \( n^* \geq 2 \). Then \( n, m \geq n^* \) and \( m > n \) implies
\[
\left| \sum_{i=1}^{m} \frac{1}{i^2} - \sum_{j=1}^{n} \frac{1}{j^2} \right| < \frac{1}{n} < \epsilon,
\]
\( \therefore \) \( \{a_n\} \) is Cauchy.

5. Section 2.6 #3
Consider the subsequence \( a_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \). We notice that
\[
1/3 + 1/4 > 1/2, \ 1/5 + 1/6 + 1/7 + 1/8 > 1/2, \text{ and in particular for each term which has the power } 2^k \text{ in the denominator, the previous } 2^{k-1} \text{ terms are all bigger than } 1/2^k \text{ and the sum (with } 2^{k-1} \text{ terms in it)}
\]
\[
\frac{1}{2^{k-1} + 1} + \frac{1}{2^{k-1} + 2} + \cdots + \frac{1}{2^k} > 2^{k-1} \left( \frac{1}{2^k} \right) = \frac{1}{2}.
\]
We sum these terms for every \( k = 1, \ldots, n \) and find \( a_{2^n} > 1 + \frac{1}{2} n \). Clearly \( \{1 + \frac{n}{2}\} \) diverges to \( +\infty \) as \( n \) goes to infinity, which implies that \( \{a_{2^n}\} \) diverges to \( +\infty \).
6. Section 2.6 #5

Let \( \{a_n\} \) be an unbounded sequence. Then for all \( M \), there exists \( n^* \) such that \( |a_n| > M \) for all \( n \geq n^* \). We construct an unbounded monotone subsequence as follows. Consider first the case that \( \{a_n\} \) is not bounded above. Let \( b_1 = a_1 \). Find \( n_1 \) such that \( |a_n| > b_1 \) for all \( n \geq n_1 \). Since the sequence is not bounded above, there exists some \( a_k > b_1 \) where \( k > 1 \). Let \( b_2 = a_k \). Since \( \{a_n\} \) is not bounded below, there exists some \( n_2 \) such that \( a_n > b_2 \) for all \( n \geq n_2 \). In particular there exists some \( a_l > b_2 \) where \( l > k \). Let \( b_3 = a_l \). Continue this way to make a monotone subsequence \( \{b_n\} \) which converges to infinity. Keep in mind that it is important to use \( n_1 \) and \( n_2 \) to assure yourself that you can construct all the \( b_i \)'s by \( a_i \)'s with increasing index. Similarly, if \( \{a_n\} \) is not bounded below, you can construct a decreasing subsequence diverging to \( -\infty \). Note that \( \{a_n\} \) must be either unbounded above or unbounded below.

Since \( \{b_n\} \) is unbounded and monotone, it diverges to infinity.

7. Section 3.1 #2 (a)

Choose \( \epsilon > 0 \). We want to show that \( |f(x) - 2| < \epsilon \). We note that

\[
\left| \frac{2x}{x-3} - 2 \right| = \left| \frac{2x}{x-3} - \frac{2x - 6}{x-3} \right| = \frac{6}{x-3} \text{ if } x > 3.
\]

But \( \frac{6}{x-3} < \epsilon \) if and only if \( x > \frac{6}{\epsilon} + 3 \). Thus for \( x > \frac{6}{\epsilon} + 3 \) we have \( \left| \frac{2x}{x-3} - 2 \right| < \epsilon \), as desired.
8. Section 3.1 #6 (a), (b)

(a)

Let \( f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \)

Clearly, \( \lim_{x \to \infty} f(x) = 0 \). It is also clear that \( f(x) \) is unbounded.

(b)

By definition, for all \( \epsilon \), there exists \( M > 0 \) such that \( |f(x) - L| < \epsilon \) if \( x > M \). Consider the function \( f(-t) \). Pick \( \epsilon > 0 \). If \( t < -M \), then \( -t > M \) and thus \( |f(-t) - L| < \epsilon \). Therefore for any \( \epsilon \) there exists \( K \) such that for \( t < K \), \( |f(-t) - L| < \epsilon \), as desired.
9. Section 3.2 #5 (a), (b)

(a) The error is in the conclusion that \(\lim_{x \to 0} x \sin \frac{1}{x} = (\lim_{x \to 0} x)(\lim_{x \to 0} \sin \frac{1}{x})\). In particular, \(\lim_{x \to 0} \sin \frac{1}{x}\) does not exist.

(b) Note: \(|\sin \frac{1}{x}| \leq 1, \forall x \in \mathbb{R} \setminus \{0\}$. Choose \(\epsilon > 0\). Let \(\delta = \epsilon\). Then \(0 < |x - 0| < \delta\) implies

\[
\left| x \sin \frac{1}{x} \right| = |x| \left| \sin \frac{1}{x} \right| < \epsilon \cdot 1 = \epsilon.
\]

Therefore, \(\lim_{x \to 0} x \sin \frac{1}{x} = 0\).
10. Section 3.2 #10

We need to show that for any $K > 0$, there exists a $\delta$ such that $\frac{1}{|f(x)|} > K$ for all $x \in D$ such that $0 < |x - a| < \delta$.

Choose $K > 0$ and $\epsilon < \frac{1}{K}$. Since $\lim_{x \to a} f(x) = 0$, there exists $\delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - 0| < \epsilon$. But then $\frac{1}{|f(x)|} > \frac{1}{\epsilon} > K$.

Therefore, $\lim_{x \to a} \frac{1}{|f(x)|} = \infty$. 